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TRIGONOMETRY.

BY

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BIBLIOTHECA BODLEIANA

Cambridge:
W. METCALFE, 12, TRINITY STREET;
London: LONGMANS, GREEN, READER, AND DYER.
1868.

183. g. 30.

PREFACE.

THE following pages are not intended to supersede oral instruction, but they contain, I hope, all that will be found necessary for mastering the elementary parts of Trigonometry. I have endeavoured to avoid introducing anything not fundamental, while attention has been paid to those points in which I have found by practical experience that beginners usually err. I presuppose that the Student has a Tutor, and put this short summary forward simply as a text-book. I have appended a system of examples, with references to the articles which they are framed to illustrate; in addition to which I have in most cases worked out one as a sample. It will, I think, be found sufficient to meet the requirements of the Honour

Little-Go, and be of use in classes of schools, as preparatory to fuller treatises.

In conclusion, I take this opportunity of thanking the Rev. H. Latham, for kindly suggesting amendments, and to H. B. Buckley, B.A., for revising the work when in manuscript.

A. STOKES.

TRINITY HALL,
CAMBRIDGE, 1868.

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A SYLLABUS OF TRIGONOMETRY.

CHAPTER I.

THE MEASUREMENT OF ANGLES BY DEGREES AND GRADES.

1. *Plane Trigonometry* comprises investigations relating to *plane* angles (Euc. I. Def. 8), such investigations being made by the aid of algebraical symbols and formulæ.

2. To compare angles we must of course have some means of measuring them. In order to measure quantities of the same denomination we must have a standard of reference or *unit*, which may be thus defined:

Def. A unit is the quantity in terms of which all other quantities of the same kind are expressed.

Thus in money matters we fix upon a certain weight of gold, and say that it shall be called one pound sterling; double that weight of gold will then be called two pounds sterling, and so on; the weight of gold therefore which we chose is our *unit*.

3. In measuring *angles* there are three methods in use; each of course having a different *unit*. They are

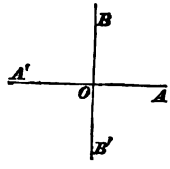
(i) The English or *sexagesimal* method.

(ii) The French or *centesimal* method.

(iii) The method of Circular measure.

The first two only will be treated of in this chapter, the third being reserved for separate discussion.

4. (i) By Euc. I. Def. 10, a right angle is defined thus: "When a straight line standing upon another straight line makes the adjacent angles equal, each of them is called a right angle."



Thus, if $\angle AOB = \angle A'OB$, each is a right angle, and by Euc. I. 15, Cor. 1,

$$AOB + BOA' + A'OB' + B'OA = 4 \text{ right angles.}$$

The English method divides a right angle into 90 equal divisions, called *degrees*, each *degree* again into 60 *minutes*, each *minute* again into 60 *seconds; one degree* being considered the unit of this system.

An angle of *eighty-four degrees, fifteen minutes, and thirty-eight seconds* is thus written:

$$84^\circ\ 15'\ 38'',$$

° to the right of 84 signifying *degrees*, ′ *minutes*, ″ *seconds*.

This method derives its name *sexagesimal* from the Latin *sexaginta*, meaning *sixty*.

5. (ii) The French method divides a right angle into 100 equal parts called *grades*, each *grade* into 100 *minutes*, each *minute* into 100 *seconds*; one grade being considered the unit in this system.

An angle of *ninety-seven grades, eighty minutes, and nineteen seconds* would be written thus:

$$97^g\ 80^{`}\ 19^{``},$$

^{g} standing for *grades*, ` for *minutes*, `` for *seconds*.

This method derives its name *centesimal* from the Latin *centum*, meaning a *hundred.*

The advantage of this system is that any angle can be at once written down as a decimal of a grade. The above, for instance, would become $97^g.8019$. So $83^g\ 71^c\ 1^{cc}$ would be at once written $83^g.7101$.

6. *To find the number of grades in an angle expressed by degrees, and vice versâ.*

Let D = the number of degrees in the angle,

G = the number of grades in the angle.

Then, since by what has been said above, 90 *degrees* are equal to 100 *grades*, each making up a right angle,

$$D : 90 :: G : 100,$$

or

$$100D = 90G;$$

therefore

$$G = \frac{100}{90} D = \frac{10}{9} D = D + \frac{D}{9},$$

$$D = \frac{90}{100} G = \frac{9}{10} G = G - \frac{G}{10}.$$

Thus to transform an angle from degrees to grades, we must express the angle in degrees and decimal of a degree, and add one-ninth of such expression to itself. This will give the angle in grades and decimal of a grade; we can then express it in grades, minutes, and seconds.

To transform an angle from grades to degrees, we must express the given angle in grades and decimal of a grade, and subtract one-tenth of such expression from itself. This will give the angle in degrees and decimal of a degree, and we can then find its value in degrees, minutes, and seconds.

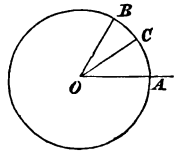
CHAPTER II.

CIRCULAR MEASURE.

7. In this chapter the third method of measuring angles will be considered.

It will be assumed that the circumferences of circles vary as their radii, and that if r be the radius of any circle, its circumference will be equal to $2\pi r$, where $\pi = 3.14159\ldots$, that is, π is the number of times that the *diameter* of a circle is contained in the circumference.

8. Let ACB be a circle of any radius OA. Mark off along the circumference an arc $AB = OA$.



Let AC be any other arc. Join OB, OC. Then (by Euc. VI. 33), since angles at the centre vary as the arcs on which they stand,

$$\frac{\angle AOC}{\angle AOB} = \frac{\text{arc } AC}{\text{arc } AB}.$$

Let arc AC be denoted by l, and OA by r; also $AB = r$ by the construction. Thus

$$\angle AOC = \frac{l}{r} \times (\angle AOB).$$

9. If now the angle AOB is an invariable angle, *i.e. if the angle subtended by an arc = radius* is invariable, whatever radius be taken, we may use it as our unit of circular measure.

Now $$\frac{\angle AOB}{4 \text{ right angles}} = \frac{\text{arc } AB}{\text{whole circumference}}.$$

i.e. $$\frac{\angle AOB}{360^\circ} = \frac{r}{2\pi r};$$

therefore $$\angle AOB = \frac{360^\circ}{2\pi} = \frac{180^\circ}{\pi} = 57^\circ.2957\ldots$$

by division, and is therefore invariable.

10. We may therefore take as our *unit of circular measure* an angle of $57^\circ.2957\ldots$, *i.e., an angle subtended by an arc = the radius.*

We have then an expression for *any* angle AOC,

$$\angle AOC = \frac{l}{r} = \frac{\text{arc}}{\text{radius}},$$

which is called the *circular measure* of $\angle AOC$; or if $AOC = \theta$, $\theta = \frac{l}{r}$; a general expression.

11. Of course if we wish to express in degrees an angle expressed in circular measure, we must multiply such an expression by the number of degrees in the unit of circular measure, *i.e.* by $57^\circ.2957\ldots$.

Thus, an angle represented in circular measure by 2 would be an angle of

$$2 \times (57^\circ.2957\ldots) = 114^\circ.5914.$$

12. To express 30°, 45°, 60°, 90°, 120°, 135°, 180°, 210°, 270°, &c. in circular measure.

$\theta = \frac{l}{r}$ is our general expression. (Art. 10).

Now if $\theta = 180°$, l = arc = $\frac{1}{2}$ circumference = $\frac{1}{2}2\pi r = \pi r$; therefore

$$\theta = \frac{\pi r}{r} = \pi.$$

That is, π is the circular measure of 180°; therefore, since

$$\frac{180°}{6} = 30°,$$

$\frac{\pi}{6}$ = circular measure of 30°,

$\frac{\pi}{4}$ = ... 45°,

$\frac{\pi}{3}$ = ... 60°,

$\frac{\pi}{2}$ = ... 90°,

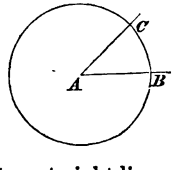
$\frac{2\pi}{3}$ = ... 120°,

$\frac{3\pi}{4}$ = ... 135°,

$\frac{7\pi}{6}$ = ... 210°,

$\frac{3\pi}{2}$ = ... 270°.

To define the circular measure of an angle.



Let AB, AC be two straight lines meeting in A. With

centre A and *any* radius describe a circle cutting AB, AC in B, C. Then, if r = radius of this circle,

$$\text{circular measure of } \angle BAC = \frac{\text{arc } BC}{r}.$$

13. *To express in degrees an angle whose circular measure is known.*

Let θ be the known circular measure, x the number of degrees in the angle. Then

$$\frac{\theta}{\pi} = \frac{x}{180};$$

therefore $$x = \frac{180}{\pi}.\theta = (57.2957\ldots)\theta,$$

as was noticed above. (Art. 11).

14. *To express in circular measure an angle which is given in degrees.*

The above formula will enable us to do this also.

Thus, using the same notation as before,

$$\theta = \pi \frac{x}{180},$$

whence x being known, θ is also known.

15. If y = the number of grades in an angle, θ its circular measure, we shall have in a similar manner

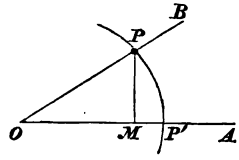
$$y = \frac{200}{\pi}\theta,$$

and $$\theta = \frac{\pi}{200}y.$$

CHAPTER III.

THE TRIGONOMETRICAL FUNCTIONS.

16. Let AOB be any angle, A suppose. In OB take



any point P. Draw PM perpendicular to OA. Then OMP is a right-angled triangle. Now

(i) $\dfrac{PM}{OP} = \dfrac{\textit{perpendicular}}{\textit{hypothenuse}}$, is called the *sine* of $\angle AOB$; *i.e.* of A, and is written $\sin A$.

(ii) $\dfrac{OM}{OP} = \dfrac{\textit{base}}{\textit{hypothenuse}}$, is called the *cosine* of A, and is written $\cos A$.

(iii) $\dfrac{OP}{PM} = \dfrac{\textit{hypothenuse}}{\textit{perpendicular}}$, is called the *cosecant* of A, and is written $\operatorname{cosec} A$.

(iv) $\dfrac{OP}{OM} = \dfrac{\textit{hypothenuse}}{\textit{base}}$, is called the *secant* of A, and is written $\sec A$.

(v) $\dfrac{PM}{OM} = \dfrac{\textit{perpendicular}}{\textit{base}}$, is called the *tangent* of A, and is written $\tan A$.

(vi) $\frac{OM}{PM} = \frac{base}{perpendicular}$, is called the *cotangent*, and is written $\cot A$.

With centre O and radius OP describe a circle cutting OA in P'.

(vii) $1 - \frac{OM}{OP} = 1 - \frac{OM}{OP'} = \frac{OP' - OM}{OP'} = \frac{P'M}{OP'}$, is called the *versine* of A, and is written $\text{vers}\,A$. The versine of A might have been defined as $1 - \cos A$.

(viii) $1 - \frac{PM}{OP}$, is called the *coversine* of A, and is written $\text{covers}\,A$; in other words, $\text{covers}\,A = 1 - \sin A$.

17. From (i) and (iii), we see that since

$$\frac{PM}{OP} \times \frac{OP}{PM} = 1, \quad \sin A \times \operatorname{cosec} A = 1;$$

therefore $\sin A = \frac{1}{\operatorname{cosec} A}$, and $\operatorname{cosec} A = \frac{1}{\sin A}$.

From (ii) and (iv),

$$\cos A \times \sec A = 1;$$

therefore $\cos A = \frac{1}{\sec A}$; $\sec A = \frac{1}{\cos A}$.

From (v) and (vi),

$$\tan A \times \cot A = 1;$$

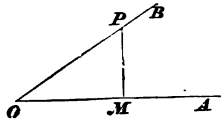
therefore $\tan A = \frac{1}{\cot A}$; $\cot A = \frac{1}{\tan A}$.

By Def. $\tan A = \frac{PM}{OM} = \frac{\frac{PM}{OP}}{\frac{OM}{OP}} = \frac{\sin A}{\cos A}$,

so $\cot A = \frac{\cos A}{\sin A}$.

18. *Def.* The *complement of an angle* is its defect from a *right angle.*

Thus, if A be any angle, $90° - A$, $100^g - A$, $\frac{\pi}{2} - A$, will express the *complement of* A according to the three methods of measurement of angles.



By our Definition, if $\angle BOA = A$,

$$\sin A = \frac{PM}{OP}.$$

Similarly $\quad \frac{PM}{OP} = \cos OPM = \cos(90° - A)$; *vide* (ii)

therefore $\quad \sin A = \cos(90° - A)$.

That is, *The sine of an angle is the cosine of its complement.*

Similarly, *The cosine of an angle is the sine of its complement.*

The tangent of an angle is the cotangent of its complement.

The cotangent of an angle is the tangent of its complement.

These may be thus expressed,

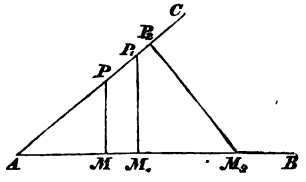
$$\sin A = \cos(90° - A); \quad \cos A = \sin(90° - A);$$

$$\tan A = \cot(90° - A); \quad \cot A = \tan(90° - A).$$

19. *Def.* The *supplement of an angle* is its defect from *two right angles.*

Thus, $180° - A$, $200^g - A$, $\pi - A$, express the supplement of A.

20. *The Trigonometrical ratios remain unchanged so long as the angle remains unchanged.*



Let BAC be any angle A. Take P, P_1 in AC, and draw PM, P_1M_1 at right angles to AB.

Also, taking P_2 in AC, draw P_2M_2 at right angles to it, meeting AB in M_2.

Then, by our definition,

$$\sin A = \frac{PM}{AP}.$$

But, by similar triangles,

$$\frac{PM}{AP} = \frac{P_1M_1}{AP_1} = \frac{P_2M_2}{AP_2}.$$

That is, the *sine* of A is the same whether we form it from the triangle APM, or AP_1M_1, or AP_2M_2.

This of course holds good for the other Trigonometrical Ratios.

21. It may be as well here to point out the modes of writing the various powers of the ratios.

The *square of* $\sin A$ is written $(\sin A)^2$, or more usually $\sin^2 A$.

The *cube of* $\sin A$ similarly as $\sin^3 A$, and so on for higher powers, and for the other ratios.

The *square root* and *cube root* of $\sin A$, are written $(\sin A)^{\frac{1}{2}}$, $(\sin A)^{\frac{1}{3}}$, or more usually $\sqrt{\sin A}$, $\sqrt[3]{\sin A}$; sometimes also $\sin^{\frac{1}{2}} A$, $\sin^{\frac{1}{3}} A$.

$\text{Sin}^2 A$, $\sin^3 A$ must be carefully distinguished from $\sin 2A$, $\sin 3A$. $\text{Sin}\, 2A$ means the *sine* of an angle, which angle is twice as large as A.

Thus, if $A = 13°$, $\sin 2A = \sin 26°$.

Similarly, $\sin 3A$ would be $= \sin 39°$.

It must also be remembered that each of the expressions $\sin A$, $\cos A$, etc., is a *ratio*, *i.e.* a *fraction*, and must be treated as an a or b would be treated in Algebra. Consequently $\sin A$ can never be written A sin, for it would have no meaning, nor can $\sin 3A$ be written $3 \sin A$, except in the case where we have to deal with an angle which would satisfy the relation $\sin 3A = 3 \sin A$.

Again, $\sin^2 A$ cannot be written either as $A \sin^2$ or $A^2 \sin$; nor do $\sin^2 A^2$ and $A^2 \sin^2$ express what is meant by $(\sin A)^2$.

22. These remarks apply also to all the ratios besides the *sine*. If we have an expression such as

$$\sin^2 A = \sin^5 A,$$

we can divide by $\sin^2 A$, *i.e.* $(\sin A)^2$, and we have

$$1 = \sin^3 A.$$

This is evident if we write our expression as

$$(\sin A)^2 = (\sin A)^5,$$

which corresponds to an algebraical equation

$$a^2 = a^5 = a^3 a^2.$$

From this we see that we may write

$$\sin^2 A \cos^3 A + \cos^2 A \sin^4 A$$

as

$$\sin^2 A \cos^2 A (\cos A + \sin^2 A).$$

23. We have seen above that

$$\frac{OM}{OP} = \cos A.$$

Hence $\quad OM = OP \cos A$;

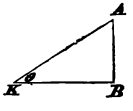
also that $\quad \dfrac{PM}{OP} = \sin A,$

or $\quad PM = OP \sin A,$

so $\quad PM = OM \tan A,$

$OM = PM \cot A$, and so on.

This way of finding an expression for one side of a triangle in terms of one of the others and some one of the ratios, should be practised.



Ex. In the triangle ABK, where $B = 90°$, if $\angle AKB = \theta$,

$$AB = BK \tan\theta,$$

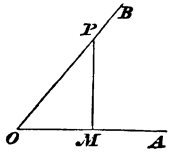
$$KB \sec\theta = AK.$$

CHAPTER IV.

CERTAIN RELATIONS AMONG THE RATIOS.

24. *To shew that, whatever angle A may be,*

$$\sin^2 A + \cos^2 A = 1.$$



Let POM be any right-angled triangle with angle $OMP = 90°$. Let $\angle POM = A$.

By Euclid I. 47,

$$OP^2 = PM^2 + MO^2;$$

therefore $$1 = \frac{PM^2}{OP^2} + \frac{MO^2}{OP^2} = \left(\frac{PM}{OP}\right)^2 + \left(\frac{MO}{OP}\right)^2,$$

or $$1 = \sin^2 A + \cos^2 A,$$

that is, *whatever A may be, this relation always holds.*

Whence $\left.\begin{array}{l}\sin^2 A = 1 - \cos^2 A \\ \text{and} \quad \cos^2 A = 1 - \sin^2 A\end{array}\right\}$, or $\begin{array}{l}\sin A = \sqrt{(1 - \cos^2 A)}, \\ \cos A = \sqrt{(1 - \sin^2 A)}.\end{array}$

That is, if we know $\sin A$ we can find $\cos A$, and *vice versâ.*

25. *To shew that*

$$\sec^2 A = 1 + \tan^2 A.$$

As before, $$OP^2 = OM^2 + MP^2;$$

therefore $$\left(\frac{OP}{OM}\right)^2 = 1 + \left(\frac{MP}{OM}\right)^2,$$

or $$\sec^2 A = 1 + \tan^2 A,$$

or, since $$\cos^2 A = \frac{1}{\sec^2 A} \text{ (Art. 17)},$$

$$\cos^2 A = \frac{1}{1 + \tan^2 A},$$

also $\sec A = \sqrt{(1 + \tan^2 A)}$, $\cos A = \dfrac{1}{\sqrt{(1 + \tan^2 A)}}$.

Again, dividing by MP^2,

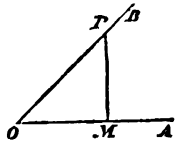
$$\left(\frac{OP}{MP}\right)^2 = \left(\frac{OM}{MP}\right)^2 + 1,$$

or $$\operatorname{cosec}^2 A = 1 + \cot^2 A,$$

and $$\sin^2 A = \frac{1}{(1 + \cot^2 A)}.$$

That is, $$\left.\begin{aligned} \operatorname{cosec} A &= \sqrt{(1 + \cot^2 A)} \\ \text{and} \quad \sin A &= \frac{1}{\sqrt{(1 + \cot^2 A)}} \end{aligned}\right\}.$$

26. Since $$\sin^2 A = 1 - \cos^2 A,$$



we have, from the above results,

$$\sin^2 A = 1 - \frac{1}{1 + \tan^2 A} = \frac{1 + \tan^2 A - 1}{1 + \tan^2 A},$$

or
$$\sin^2 A = \frac{\tan^2 A}{1 + \tan^2 A},$$
$$\sin A = \frac{\tan A}{\sqrt{(1 + \tan^2 A)}}.$$

We have thus expressed both $\sin A$ and $\cos A$ in terms of $\tan A$. We also know that

$$\cot A = \frac{1}{\tan A}, \quad \sec A = \frac{1}{\cos A}, \quad \operatorname{cosec} A = \frac{1}{\sin A}.$$

We have therefore only to express $\operatorname{vers} A$, $\operatorname{covers} A$, in terms of $\tan A$, and we shall have expressed all the other ratios in terms of $\tan A$. The *coversine* is so seldom used that it will be enough to shew it for the *versine*,

$$\operatorname{versin} A = 1 - \cos A = 1 - \frac{1}{\sqrt{(1 + \tan^2 A)}}.$$

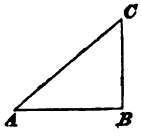
27. By similar methods, if we have any trigonometrical ratio given, we can find all the others from it. Thus, if

$$\sin\theta = \tfrac{1}{2},$$
$$\cos\theta = \sqrt{(1 - \sin^2\theta)} = \sqrt{(1 - \tfrac{1}{4})} = \pm \frac{\sqrt{(3)}}{2};$$
$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{1}{2}}{\frac{\sqrt{(3)}}{2}} = \pm \frac{1}{\sqrt{(3)}}. \quad \text{(Art. 17).}$$

CHAPTER V.

THE RATIOS OF CERTAIN ANGLES.

28. *To find the Trigonometrical ratios for* 30°, 45°, 60°; that is, *of* $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$.



Let BAC be a right-angled triangle with right angle at B.

Let the angle $BAC = 45°$; therefore also the angle $ACB = 45°$ (Euc. I. 32), and

$$AB = BC.$$

By (Euclid I. 47)

$$AC^2 = AB^2 + BC^2 = 2AB^2,$$

whence
$$\left(\frac{AB}{AC}\right)^2 = \frac{1}{2},$$

or
$$\frac{AB}{AC} = \cos 45° = \frac{1}{\sqrt{(2)}} = \cos\frac{\pi}{4}.$$

Similarly
$$AC^2 = 2BC^2,$$

and
$$\frac{BC}{AC} = \frac{1}{\sqrt{(2)}} = \sin 45° = \sin\frac{\pi}{4}.$$

Hence $$\tan 45^\circ = \tan\frac{\pi}{4} = \frac{\frac{1}{\sqrt{(2)}}}{\frac{1}{\sqrt{(2)}}} = 1,$$

$$\cot 45^\circ = \frac{1}{\tan 45^\circ} = \frac{1}{1} = 1,$$

$$\sec 45^\circ = \frac{1}{\cos 45^\circ} = \frac{1}{\frac{1}{\sqrt{(2)}}} = \sqrt{(2)},$$

$$\operatorname{cosec} 45^\circ = \frac{1}{\sin 45^\circ} = \sqrt{(2)}.$$

29. Without a figure this may be shewn thus:
We know that if A be any angle,

$$\sin^2 A + \cos^2 A = 1.$$

Now we also know that

$$\sin A = \cos(90^\circ - A) \text{ (Art. 18)},$$

so that $$\sin 45^\circ = \cos(90^\circ - 45^\circ) = \cos 45^\circ,$$

and the above becomes

$$2\sin^2 45^\circ = 1;$$

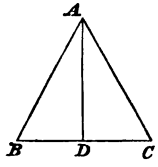
therefore $$\sin 45^\circ = \frac{1}{\sqrt{(2)}} = \cos 45^\circ \text{ as before.}$$

30. Let ABC be an equilateral triangle.
Each of its angles $= 60^\circ$, *i.e.* $B = 60^\circ$.
Draw AD perpendicular to BC.
Then (by Euc. I. 26)

$$BD = DC = \tfrac{1}{2}AB,$$

since $$AB = BC = AC.$$

Hence $$\cos 60^\circ = \frac{BD}{AB} = \frac{\frac{1}{2}AB}{AB} = \frac{1}{2},$$



$$\sin 60^\circ = \sqrt{(1 - \cos^2 60^\circ)} = \sqrt{(1 - \tfrac{1}{4})} = \frac{\sqrt{(3)}}{2}.$$

Hence $$\tan 60^\circ = \frac{\frac{\sqrt{(3)}}{2}}{\frac{1}{2}} = \sqrt{(3)}.$$

Now $\cos 60^\circ = \sin(90^\circ - 60^\circ) = \sin 30^\circ$ (Art. 18);

therefore $$\sin 30^\circ = \frac{1}{2},$$

also $\cos 30^\circ = \sin(90^\circ - 30^\circ) = \sin 60^\circ$ (Art. 18);

therefore $$\cos 30^\circ = \frac{\sqrt{(3)}}{2},$$

and $$\tan 30^\circ = \frac{1}{\sqrt{(3)}}.$$

The sine, cosine, and tangent of 30°, 45°, 60° should be remembered.

In Art. 28 we found

$$\sin 45^\circ = \frac{1}{\sqrt{(2)}}.$$

This, if expressed as a decimal, would be

$$\sin 45^\circ = .70710, \text{ so } \cos 30^\circ = .86602.$$

Tables have been formed of all the Trigonometrical Functions, calculated to *five* places of decimals, and are called "Tables of Natural Sines, Cosines, &c." It is found that to five places of decimals the variation between the

natural sines, &c. of *angles differing only by one degree* is proportional to the variation in the angle.

Suppose we are asked to find $\sin 46° 35' 0''$.

In the tables we have

$$\sin 46° = .71934, \quad \sin 47° = .73135.$$

The difference for 1°, *i.e.* 60′, is therefore

$$.73135 - .71934 = .01201.$$

Let x be the required difference for 35′; by what has been said, we have this proportion,

$$60' : 35' :: .01201 : x,$$

$$x = \frac{.01201 \times 35}{60} = \frac{.01201 \times 7}{12} = \frac{.08407}{12} = .007 \text{ nearly,}$$

and $$\sin 46° 35' = .71934 + .007 = .72634.$$

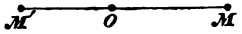
CHAPTER VI.

APPLICATION OF ALGEBRAICAL SIGNS.

31. In Euclid we have to deal with angles less than two right angles.

In Trigonometry however we are not limited in any way as to the size of the angles. In order more conveniently to determine the various relations which we shall meet, certain conventions have been used.

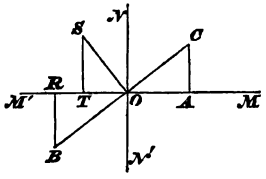
32. Let O be a fixed point, MOM' a fixed line passing through it.



It is found convenient to call all lines measured to the *right* of O, *i.e.* from O to M, *positive*; those from O towards M', *negative*.

Thus if OM represent a distance 2. OM', if measured equal to OM, will be represented by -2.

Again, through O draw a line NON' at right angles to MOM'.



We again, for convenience, shall call all lines measured from O *above* MOM', *positive*; all *below*, *negative*.

Thus if we take points C, S, B within the angles MON, NOM', $M'ON'$ respectively, and complete the figure,

AC is + being *above* $M'OM$,

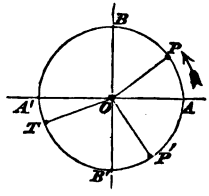
OA is + being to *the right* of NON'.

But OT is − being to *the left* of NON',

while ST is +.

Again, BR, OR are both negative.

33. These conventions we shall find sufficient for lines. We shall now proceed to angles.



Suppose O a fixed point, draw $A'OA$, $B'OB$ at right angles to each other through it.

Let OP revolve from an initial position OA, and come to the position P, by revolving from *right to left*, *i.e.* in a direction *contrary* to the hands of a watch.

All angles generated by OP revolving from *right to left* we shall call *positive;* those by OP revolving in direction from A towards B', *i.e. as* the hands of a watch, *negative.*

Thus AOP is a positive angle.

If now OP revolves on through B, A', B' to P', AOP' the angle traced out will still be positive.

But if OP start from OA and revolve through P', A', B to P, then AOP' is negative, as also the $\angle AOP$ ($>180°$).

It will be observed that each of the angles AOB, BOA', $A'OB'$, $B'OA$ is a right angle, and thus the whole circle is

divided into four equal compartments. These are called *quadrants*, and are numbered in order according to their succession in the *positive* direction for angles, *i. e.*

AOB is the first quadrant,

BOA' is the second,

$A'OB'$ is the third,

$B'OA$ is the fourth.

Thus AOP is an angle in the *first quadrant*, AOT ($> 180°$) an angle in the *third quadrant*.

34. We have said that we are not to be limited as to the size of our angles.

Hence if we call the angle AOP, A, and suppose OP to revolve from OA through all the quadrants round again to OP, moving always in the positive direction, we have an angle $(360° + A)$.

If through two *revolutions* (*i. e.* twice through all the quadrants) and on to OP we have an angle $720° + A$, and so on as large as we like.

So, again, if OP revolves from OA in the negative direction once round and beyond to OP', we have an angle

$$-(360° + A') \text{ if } OAP' = A'.$$

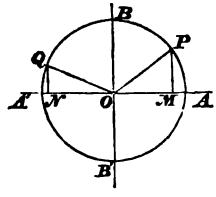
35. Let OP, OQ be two positions of the revolving line.

Then $$\sin POA = \frac{PM}{OP}.$$

If now OP revolve through 360° it will again be at OP, and we have

$$\sin(360° + A) = \frac{PM}{OP}, \text{ as before,}$$

for OA is the original base, OP is the hypothenuse, PM is a perpendicular from any point in OP upon OA.



Similarly $$\sin(720° + A) = \frac{PM}{OP}.$$

Now the fundamental triangle for the angle AOQ is found by dropping a perpendicular from Q upon AOA' the base produced. Hence

$$\cos AOQ = -\frac{ON}{OQ},$$

for ON is to the *left* of O, and therefore negative.

So $$\cos(360° + AOQ) = -\frac{ON}{OQ} = \cos(720° + AOQ).$$

Hence we see that *the values of the trigonometrical ratios recur after every complete revolution.*

36. A complete revolution is equivalent to moving through 360°, *i.e.* 2π, as expressed in circular measure, for the C. M. of 180° was shewn to be π (Art. 12); so

$$\text{C. M. of } 360° + A = 2\pi + a, \quad \text{of } 720° + A = 4\pi + a,$$

where a is the value of A expressed in C. M., *i.e. generally* $2n\pi + a$, where n is any positive integer or zero.

Hence from the last article

$$\sin a = \sin(2n\pi + a), \quad \cos a = \cos(2n\pi + a).$$

Similar results hold in the case of angles formed by revolutions in a negative direction. Thus

$$\cos a = \cos\{-(2n\pi + a)\}.$$

37. If therefore we know that an angle θ has a certain *sine, cosine, tangent,* &c., we know that any angle given

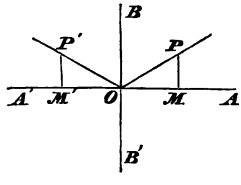
by $\pm(2n\pi + \theta)$ will also have the same *sine, cosine, tangent,* &c. as θ.

38. We have shewn (Art. 18) that $\sin(90° - A) = \cos A$ and that similar relations hold among other of the ratios. We will now show that

$$\sin A = \sin(180° - A),$$
$$\cos A = -\cos(180° - A).$$

Let the angle $POA = A$.

At the point O in $A'O$ make the angle $P'OA' = POA$.



Then $\quad MOP' = 180° - A'OP' = 180° - A.$

Take any point P in OP, draw PM perpendicular to OA, and let OP revolve from right to left, *i.e.* in the *positive* direction, so that $OP' = OP$.

Draw $P'M'$ at right angles to OA'.

Then, by construction, the triangles POM, $P'OM'$ are equal in all respects (Euc. I. 26).

Now $\sin(180° - A) = \sin P'OA = \dfrac{P'M'}{OP'} = \dfrac{PM}{OP}$,

for OP the revolving line is considered to be always positive, and PM being measured above AOA' is *positive*, and equal in magnitude to PM. Hence

$$\sin(180° - A) = \frac{PM}{OP} = \sin A.$$

Again, $$\cos(180^\circ - A) = \frac{OM'}{OP'}.$$

Now $OP' = OP$ as before, but OM', though equal in magnitude to OM, is of different sign, being measured to the left of BOB', and therefore *negative*. Hence

$$\cos(180^\circ - A) = -\frac{OM}{OP} = -\cos A.$$

39. From the last article we see that

$$\tan(180^\circ - A) = \frac{\sin(180^\circ - A)}{\cos(180^\circ - A)} = \frac{\sin A}{-\cos A} = -\tan A,$$

or we might prove this at once from the figure. Thus,

$$\tan(180^\circ - A) = \frac{P'M'}{OM'} = \frac{PM}{-OM} = -\tan A,$$

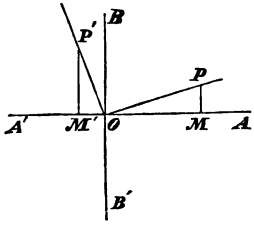
so $$\sec(180^\circ - A) = \frac{1}{\cos(180^\circ - A)} = \frac{1}{-\cos A} = -\sec A,$$

$$\operatorname{cosec}(180^\circ - A) = \frac{1}{\sin(180^\circ - A)} = \frac{1}{\sin A} = \operatorname{cosec} A,$$

$$\cot(180^\circ - A) = \frac{1}{\tan(180^\circ - A)} = \frac{1}{-\tan A} = -\cot A,$$

$$\operatorname{versin}(180^\circ - A) = 1 - \cos(180^\circ - A) = 1 + \cos A.$$

40. *To connect the trigonometrical ratios of an angle with those of the angle formed by adding* 90° *to it.*



Let $POM = A$. Draw PM at right angles to OA.
Let OP revolve through 90° to OP'.
Then $P'OM = 90° + A$.
Draw $P'M'$ at right angles to OA'.
Then, by Geometry,

$$OM' = PM, \quad P'M' = OM \textit{ numerically.}$$

Now $$\sin(90° + A) = \frac{P'M'}{OP'} = \frac{+OM}{OP} = \cos A,$$

$$\cos(90° + A) = \frac{OM'}{OP'}, \text{ and since } OM' \text{ is } \textit{negative},$$

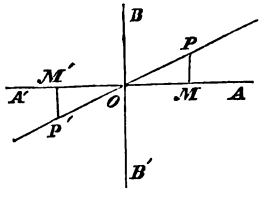
$$= \frac{-PM}{OP} = -\sin A,$$

$$\tan(90° + A) = \frac{\cos A}{-\sin A} = -\cot A,$$

$$\cot(90° + A) = \frac{-\sin A}{\cos A} = -\tan A.$$

41. *To connect the trigonometrical ratios of an angle and the angle formed by adding* 180° *to the angle.*

Let $POM = A$.
Produce PO to P'.
Make $OP' = OP$.
Draw $P'M'$ at right angles to OA'.



Then the triangles POM, $P'OM'$ are equal in all respects, and $AOP' = 180° + A$ measuring from right to left in the *positive* direction; and we have

$$\sin(180° + A) = \frac{P'M'}{OP'} = \frac{-PM}{OP} = -\sin A,$$

$$\cos(180° + A) = \frac{OM'}{OP'} = \frac{-OM}{OP} = -\cos A,$$

and similar relations may be shewn to hold good for the other ratios.

42. Suppose we meet with such an angle as $540° + A$, and wish to connect its *sine* with that of A.

The angle may be thus written, $360° + (180° + A)$.

Now, by Art. 35, the ratios recur after every complete revolution, *i.e.* after every addition (or subtraction) of 360°.

Hence $\sin(540° + A) = \sin(180° + A) = -\sin A$ by the last article.

Consequently, when dealing with large angles we need only concern ourselves with finding how many complete revolutions, *i.e.* how many times 360° are contained in it, and then proceed to find the value of the required ratio of the remainder (if any). If there be no remainder, the trigonometrical ratio of an angle of 0° is the one required.

43. By continuing the proofs of the last three articles for the fourth quadrant, it will be seen that throughout the quadrants the *algebraical sign* of each ratio varies. Thus $\sin A$ in the first quadrant is *positive*.

$\sin(180° + A)$ in the third $= \frac{P'M'}{OP'}$ (fig., Art. 41) and is *negative*, for $P'M'$ is negative.

Variations of this kind will be shewn to hold good in the case of the other ratios. At present we shall only remark that since we have

$$\sin^2 A = 1 - \cos^2 A,$$

$$\sin A = \pm \sqrt{(1 - \cos^2 A)},$$

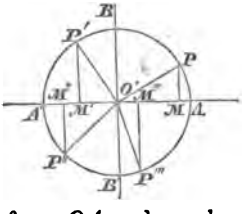
we must have some guide in any particular case to shew us which *sign* is to be taken.

CHAPTER VII.

ON TRACING THE CHANGES OF THE TRIGONOMETRICAL FUNCTIONS.

44. *To trace the change in sign and magnitude of the sine of an angle as the angle varies.*

Let AOA', BOB' be our two lines mutually at right angles, dividing the plane of the paper into four quadrants.



Let OP start from OA and revolve in the positive direction.

Then $$\sin POM = \frac{PM}{OP}.$$

In the *first quadrant* PM is positive, and therefore the *sine* is positive, *for* OP *is of invariable sign.*

As OP revolves from OA to OB, PM *increases* from 0 to OB, *i.e.* from 0 to OP.

Hence $\sin POM$ varies from $\frac{0}{OP}$ to $\frac{OP}{OP}$, *i.e.* from 0 to 1, and is *positive.*

In the *second quadrant* PM is still *positive,* and therefore the *sine* positive.

As OP revolves from OB to OA', PM decreases from OP to 0.

Hence $\sin POM$ varies from $\frac{OP}{OP}$ to $\frac{0}{OP}$, *i.e.* from 1 to 0 and is *positive.*

In the *third quadrant* PM is *negative,* being below AOA', and therefore the *sine* is *negative.*

As OP revolves from OA' to OB', PM *increases* from 0 to OB', *i.e.* from 0 to OP, being negative; so that we have $\sin POM$ varying from $\frac{0}{OP}$ to $-\frac{OP}{OP}$, *i.e.* from 0 to -1 and *negative.*

In the *fourth quadrant* PM is still negative, and therefore the *sine* is *negative.*

As OP revolves from OB' to OA, PM *decreases* from OB', *i.e.* OP to 0, being negative, so that we have $\sin POM$ varying from $-\frac{OP}{OP}$ to $\frac{0}{OP}$, *i.e.* from -1 to 0, and *negative.*

45. These results may be thus tabulated:

The sine varies in quadrant

1	from 0 to + 1 and is positive,
2	from + 1 to 0 .....,
3	from 0 to − 1 and is negative,
4	from − 1 to 0 ..... .

It may also be noticed that

$$\sin 0^\circ = 0,$$

$$\sin 90^\circ = \sin\frac{\pi}{2} = +1,$$

$$\sin 180^\circ = \sin\pi = 0,$$

$$\sin 270^\circ = \sin\frac{3\pi}{2} = -1,$$

$$\sin 360^\circ = \sin 2\pi = 0.$$

46. *To trace the change in sign and magnitude of the cosine as the angle varies.*

Using the same figure as in the proof for the *sine*, we have

$$\cos POM = \frac{OM}{OP},$$

and therefore depends on the sign, and magnitude of OM, since OP is of invariable sign and magnitude.

In the *first quadrant* OM is *positive*, and therefore the cosine is positive.

As OP revolves from OA to OB, OM *decreases* from OA, *i.e.* OP, to 0.

Hence $\cos POM$ varies from $\frac{OP}{OP}$ to $\frac{0}{OP}$, *i.e.* from 0 to 1 and *positive*.

In the *second quadrant* OM is *negative*, being measured to the left, and therefore the cosine *negative*.

As OP revolves from OB to OA', OM *increases* from 0 to OA', *i.e.* OP, being *negative*, so that we have $\cos POM$ varying from $\frac{0}{OP}$ to $-\frac{OP}{OP}$, *i.e.* from 0 to -1 and *negative*.

In the *third quadrant* OM is still *negative*, and the cosine therefore negative.

As OP revolves from OA' to OB', OM *decreases* from OP to 0, being negative, so that we have $\cos POM$ varying from $-\frac{OP}{OP}$ to $\frac{0}{OP}$, *i.e.* from -1 to 0, and *negative*.

In the *fourth quadrant* OM is *positive*, and the cosine therefore positive.

As OP revolves from OP' to OA, OM increases from 0 to OA, *i.e.* OP, being positive, so that we have $\cos POM$ varying from $\frac{0}{OP}$ to $\frac{OP}{OP}$, *i.e.* from 0 to 1, and positive.

47. These results may be thus tabulated:
The cosine varies in quadrant

1	from + 1 to 0 and is positive,
2	from 0 to − 1 and is negative,
3	from − 1 to 0 .....,
4	from 0 to + 1 and is positive.

The following results may also be noticed:

$$\cos 0^\circ = +1,$$

$$\cos 90^\circ = \cos\frac{\pi}{2} = 0,$$

$$\cos 180^\circ = \cos\pi = -1,$$

$$\cos 270^\circ = \cos\frac{3\pi}{2} = 0,$$

$$\cos 360^\circ = \cos 2\pi = +1.$$

48. *To trace the change in sign and magnitude of the tangent as the angle varies.*

We shall only write down the results in this case, which is proved at length after the manner of the two preceding articles.

The tangent varies in quadrant

1	from 0 to + ∞ and is positive,
2	from + ∞ to 0 and is negative,
3	from 0 to ∞ and is positive,
4	from ∞ to 0 and is negative.

49. The appearance of the symbol for *infinity* may perhaps perplex the student, seeing that it does not occur with the *sine* and *cosine*. If however he will draw an angle very nearly 90°, he will see that by making the *base* small enough, the ratio of the *perpendicular* to the base may be

made as large as he pleases, *i.e.* the tangent of the angle (vide Def., Art. 16) is very large, and may be made indefinitely so, the nearer we approach to 90°. This is expressed by saying that

$$\tan 90° = \infty .$$

In our figure

$$\tan POM = \frac{PM}{OM}.$$

Now, at 90°, $OM = 0,\ PM = OP,$

$$\tan POM = \frac{OP}{0} = \infty ,$$

according to the usual interpretation of that symbol.

We have

$$\tan\ 0° = 0,$$

$$\tan\ 90° = \tan\frac{\pi}{2} = +\infty ,$$

$$\tan 180° = \tan \pi = 0,$$

$$\tan 270° = \tan\frac{3\pi}{2} = \infty ,$$

$$\tan 360° = \tan 2\pi = 0.$$

50. The *secant, cosecant, cotangent,* &c. may be treated in the same way, or derived from the preceding articles.

Thus $$\sec 180° = \frac{1}{\cos 180°} = \frac{1}{-1} = -1,$$

$$\cot 90° = \frac{1}{\tan 90°} = \frac{1}{+\infty} = 0.$$

51. From what has been said it may be seen that the algebraical signs of the ratios vary in the different quadrants.

Consequently, if we know in which quadrant an angle lies, we can determine the *sign* to be given to its *sine*, *cosine*, &c., and thus, in such expressions as

$$\sin A = \pm \sqrt{(1 - \cos^2 A)},$$

we are no longer in doubt as to the proper *sign*.

If A lie in the third quadrant, $\sin A$ is negative; therefore we write

$$\sin A = -\sqrt{(1 - \cos^2 A)}.$$

52. If the student will refer back to Arts. 44–50 he will see that we have the means of obtaining every result given in the following table:

	0	30°	45°	60°	90°	120°	135°	150°	180°
sine	0	$\frac{1}{2}$	$+\frac{1}{\sqrt{(2)}}$	$\frac{\sqrt{(3)}}{2}$	1	$\frac{\sqrt{(3)}}{2}$	$\frac{1}{\sqrt{(2)}}$	$\frac{1}{2}$	0
cosine	1	$\frac{\sqrt{(3)}}{2}$	$\frac{1}{\sqrt{(2)}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{(2)}}$	$-\frac{\sqrt{(3)}}{2}$	-1
tangent	0	$\frac{1}{\sqrt{(3)}}$	1	$\sqrt{(3)}$	∞	$-\sqrt{(3)}$	-1	$-\frac{1}{\sqrt{(3)}}$	0
cotangent	∞	$\sqrt{(3)}$	1	$\frac{1}{\sqrt{(3)}}$	0	$-\frac{1}{\sqrt{(3)}}$	-1	$-\sqrt{(3)}$	∞
secant	1	$\frac{2}{\sqrt{(3)}}$	$\sqrt{(2)}$	2	∞	-2	$-\sqrt{(2)}$	$-\frac{2}{\sqrt{(3)}}$	-1
cosecant	∞	2	$\sqrt{(2)}$	$\frac{2}{\sqrt{(3)}}$	1	$\frac{2}{\sqrt{(3)}}$	$\sqrt{(2)}$	2	∞

It will be of great use to the student if he will verify some of those results in this table, which are new to him, with the help of this present chapter. The table is not intended to be remembered as a whole, it will be found

sufficient to recollect the *sine* and *cosine* for 0°, 30°, 45°, 60°, 90°, the tangent will thus be known, and the other ratios easily deduced.

53. The following facts are worthy of notice:

The *sine* is positive in quadrants 1, 2; negative in 3, 4,

cosine ..... 1, 4; 2, 3,

tangent ..... 1, 3; ..... 2, 4.

Also, *that neither sine nor cosine are ever numerically greater than unity.*

CHAPTER VIII.

ANGLES WITH GIVEN TRIGONOMETRICAL RATIOS.

54. From what has been shewn we know that all the values recur after one or more complete revolutions (Art. 35). Now if we have an equation

$$\sin A = \frac{1}{2},$$

we know that A may be 30° or 150° (Art.38), and by what has just been said it may be

$$360° + 30° = 390° \text{ or } 360° + 150° = 510°,$$

or

$$720° + 30° \text{ or } 720° + 150°,$$

or, by Art. 36, any angle given by the expression

$$2n\pi + \frac{\pi}{6}, \text{ or } 2n\pi + \left(\pi - \frac{\pi}{6}\right),$$

where n is any integer, positive or negative, or zero.

We shall now shew how we may find *one* formula to contain all the angles given by such an equation.

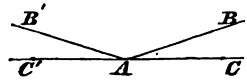
55. *To find a general expression for all angles having a given sine.*

Let BAC be the least *positive* angle having the given *sine.*

Let BAC expressed in circular measure be called a.

At A in AC' make $C'AB' = BAC$, then $CAB' = \pi - a$.

First, taking positive angles. Of such angles the only ones which satisfy the given *sine* are a, $\pi - a$, and angles formed by adding even multiples of π to these angles,



i. e. by adding complete revolutions. Such angles will be given by

$$2\pi + a, \quad 2\pi + \pi - a,$$

and $$2n\pi + a, \quad (2n + 1)\,\pi - a \text{ .....(i)},$$

where n is zero or a positive integer.

Again, taking *negative* angles. Of such angles the only ones having the given sine are

$$-(\pi + a), \quad -(2\pi - a),$$

and angles formed by adding even multiples of π to these in a negative direction, *i. e.* angles given by

$$-2n\pi - (\pi + a), \quad -2n\pi - (2\pi - a),$$

or $$-(2n + 1)\,\pi - a, \quad -2\,(n + 1)\,\pi + a \text{ ..... (ii)}.$$

From (i) and (ii) it will be seen that when we have an even multiple of π we have $+a$ following; where an odd multiple of π we have $-a$.

This therefore must be provided for in any general formula comprehending both. The formula

$$n\pi + (-1)^n\, a$$

will be found to do this, and on trial we shall find that it contains all the angles found above and no others, *n being any integer, positive or negative, or zero.*

56. In any case, therefore, where we have given the *sine* of any angle, we must find the least positive angle with the given sine, express this in circular measure (a), and all the angles will be given by the formula

$$n\pi + (-1)^n\, a.$$

This will also *hold for the cosecant.*

57. *To find an expression for all angles with the same cosine.*

Using the same figure, let BAC be the least positive angle with the given cosine, let its circular measure be denoted by a.

The only positive angles with the given cosine are a, $2\pi - a$, and any angles formed by adding even multiples of π to these, *i. e.* angles given by

$$2n\pi + a, \quad 2n\pi + 2\pi - a,$$

or

$$2n\pi + a, \quad 2(n+1)\pi - a \text{ .....(i)},$$

where n is zero or a positive integer.

The only negative angles with the given cosine are $-a$, $-(2\pi - a)$, and angles formed by adding even multiples of π to such angles, *i. e.* angles given by

$$-2n\pi - a, \quad -2n\pi - (2\pi - a),$$

that is,

$$-2n\pi - a, \quad -2(n+1)\pi + a \text{ .....(ii)}.$$

From (i) and (ii) it will be seen that we have only even multiples of π, with both $+a$ and $-a$ following.

The formula

$$2n\pi \pm a,$$

where n is any integer, positive or negative, or zero, will be found to include all the angles found above, and no others.

This will *also hold for the secant.*

58. *To find an expression for all angles having a given tangent.*

Using the same figure as before, the only positive angles will be given by a, $\pi + a$, and even multiples of π added to these, *i. e.* by

$$2n\pi + a, \quad (2n+1)\pi + a \text{ ..... (i)}.$$

The only negative angles will, in like manner, be given by

$$-2n\pi - (\pi - a), \quad -2n\pi - (2\pi - a),$$

i. e. by

$$-(2n+1)\pi + a, \quad -2(n+1)\pi + a. \text{ .....(ii)}.$$

Hence one formula must give odd and even multiples of π followed by $+ a$. The formula

$$n\pi + a$$

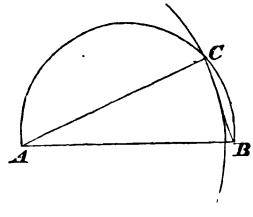
will on trial be found to give the angles found above, and no others.

This holds for *the cotangent also.*

59. *To construct an angle with a given sine.*

Let the given sine $= a$.

Take AB to represent unity, and on AB describe a semi-circle ACB. With centre A and radius $= a$, describe a circle cutting the semi-circle in C.



(It must always cut it, since, by Art. 53, the sine can never be greater than unity, *i.e.* a can never be greater than AB). Join AC, CB.

Then ACB is evidently a right angle, Euc. III. 31,

$$\frac{AC}{AB} = a = \sin CAB,$$

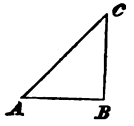
and CAB is the required angle.

60. *To construct an angle having a given cosine.*

The same construction as above being used, the angle ABC will be the angle required.

61. *To construct an angle having a given tangent.*

Let the given tangent $= a$.



Take AB to represent unity.

Draw BC at right angles to AB and $= a$.

Join AC.

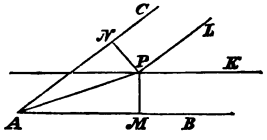
Then $$\tan CAB = \frac{CB}{AB} = a,$$

and CAB is the angle required.

If the cotangent were given $= a$, ACB would be the angle.

62. *To divide a given angle into two parts whose sines shall be in a given ratio.*

Let BAC be the angle, $\frac{m}{n}$ the given ratio.



Draw a straight line KP parallel to AB at a distance m from it, and a straight line LP parallel to AC at a distance n from it, P being the point of intersection of these lines.

Draw PM, PN at right angles to AB, AC.

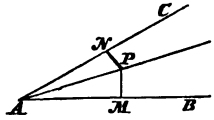
Then $$\frac{\sin PAM}{\sin PAN} = \frac{\dfrac{PM}{AP}}{\dfrac{PN}{AP}} = \frac{PM}{PN} = \frac{m}{n},$$

and PAM, PAN are the required parts.

63. *To divide a given angle into two parts whose cosines shall be in a given ratio.*

Let BAC be the given angle, $\frac{m}{n}$ the given ratio.

In AB make $AM = m$, and in AC, $AN = n$.



Draw perpendiculars to AB, AC through M, N, meeting in P. Join AP.

$$\frac{\cos PAM}{\cos PAN} = \frac{\dfrac{AM}{AP}}{\dfrac{AN}{AP}} = \frac{AM}{AN} = \frac{m}{n}.$$

64. *To divide a given angle into two parts whose tangents shall be in a given ratio.*

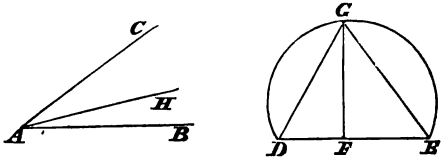
Let BAC be the given angle, $\frac{m}{n}$ the given ratio.

Take any straight line DE, divide it in F so that

$$DF : FE = m : n.$$

Upon DE describe a segment of a circle containing an angle $= BAC$.

Draw FG at right angles to DE, meeting the circumference in G. Join GD, GE.



At the point A make the angle BAH = angle DGF; therefore the remaining angle HAG = angle FGE, since $BAC = DFE$ by construction (Euc. III. 33).

$$\text{Now } \frac{\tan BAH}{\tan HAC} = \frac{\tan DGF}{\tan FGE} = \frac{\dfrac{FD}{FG}}{\dfrac{FE}{FG}} = \frac{FD}{FE} = \frac{m}{n}.$$

Hence BAH, HAC are the parts required.

65. If the cotangents were required to be in the ratio $\frac{m}{n}$, we should have

$$\frac{\cot FGE}{\cot FGD} = \frac{\dfrac{FG}{FE}}{\dfrac{FG}{FD}} = \frac{FD}{FE} = \frac{m}{n} = \frac{\cot HAC}{\cot BAH},$$

and HAC, BAH are the required parts.

66. Before proceeding to the expansions of sines, cosines, &c. of compound angles, it may be well to call the notice of the student to the fact, that in Trigonometry all algebraical devices and rules are allowable, provided that nothing be done contrary to what is stated in Arts. 21, 22. Hence, if we have an expression such as $\tan\{x + y - (z + x)\}$, it may be written as $\tan(x + y - z - x)$, *i.e.* $\tan(y - z)$; so $\tan x$ may be written $\tan\{(x - z) + z\}$, and so on.

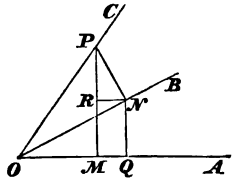
Thus $\cos(3a + 4a) = \cos 7a$, $\cos\left(\frac{6a}{3}\right) = \cos 2a$.

CHAPTER IX.

TRIGONOMETRICAL FUNCTIONS OF TWO ANGLES.

67. *To express the sine of the sum of two angles in terms of sines and cosines of the angles themselves.*

Let $\angle AOB = A, \ \angle BOC = B.$



Then $\angle AOC = A + B.$

In OC take any point P, draw PM, PN perpendiculars upon OA, OB respectively.

From N draw NQ, NR at right angles to OA, PM.

Then in the triangle PRN, since

$$\angle PRN = 90°,$$

$$\angle RPN = 90° - \angle PNR = \angle RNO = A.$$

Hence $RN = NP \sin RPN = NP \sin A,$

and $PR = NP \cos A.$

Now $\sin(A + B) = \dfrac{PM}{OP} = \dfrac{PR + RN}{OP} = \dfrac{PR + NQ}{OP}.$

But $PR = NP \cos A = OP \sin B \cos A,$

$NQ = ON \sin A = OP \cos B \sin A;$

$$\text{therefore } \sin(A+B) = \frac{OP \sin A \cos B + OP \cos A \sin B}{OP}$$

$$= \sin A \cos B + \cos A \sin B.$$

68. *To express in a similar manner the cosine of the sum of two angles.*

Using the above figure

$$\cos(A+B) = \frac{OM}{OP} = \frac{OQ - MQ}{OP} = \frac{OQ - RN}{OP}$$

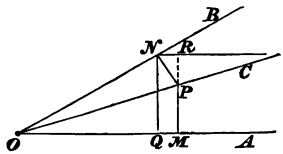
$$= \frac{ON \cos A - NP \sin A}{OP}$$

$$= \frac{OP \cos B \cos A - OP \sin B \sin A}{OP}$$

$$= \cos A \cos B - \sin A \sin B.$$

69. *To express the sine and cosine of the difference of two angles in a similar manner.*

Let A and B be the given angles, and suppose A the greater.



Let $\angle AOB = A$, $\angle BOC = B$, then $\angle AOC = A - B$.

In OC take any point P, draw PM, PN at right angles to OA, OB, from N draw NQ at right angles to OA, and NR to MP produced.

In the triangle NPR, $\angle PRN = 90°$.

The angle $NPR = 90° - \angle PNR = \angle RNB = A$.

Hence $\quad PR = PN\cos A,\ \ NR = PN\sin A,$

$$\sin(A-B) = \frac{PM}{OP} = \frac{RM-PR}{OP} = \frac{NQ-PR}{OP}$$

$$= \frac{ON\sin A - PN\cos A}{OP}$$

$$= \frac{OP\cos B\sin A - OP\sin B\cos A}{OP}$$

$$= \sin A\cos B - \cos A\sin B.$$

70. $$\cos(A-B) = \frac{OM}{OP} = \frac{OQ+NR}{OP}$$

$$= \frac{ON\cos A + PN\sin A}{OP}$$

$$= \frac{OP\cos B\cos A + OP\sin B\sin A}{OP}$$

$$= \cos A\cos B + \sin A\sin B.$$

71. In constructing the above figures, it will be of some help to the student, if he will remember that the point P is taken in the last line which is drawn, when the angles are drawn in order. Thus to form A we first draw OA, then OB, and, lastly, to form B, OC. Hence P must be taken in OC.

The student is advised to draw the figures for himself, and work out the cases in which the angle A is greater than 90° and less than 180°.

We give the four formulæ just proved, together, so that they may be committed to memory.

$$\sin(A+B) = \sin A\cos B + \cos A\sin B \ldots\ldots(1),$$
$$\sin(A-B) = \sin A\cos B - \cos A\sin B \ldots\ldots(2),$$
$$\cos(A+B) = \cos A\cos B - \sin A\sin B \ldots\ldots(3),$$
$$\cos(A-B) = \cos A\cos B + \sin A\sin B \ldots\ldots(4).$$

72. *To express the sum or difference of the sines or cosines of two angles, in terms of the sines or cosines of the semi-sum or semi-difference of the angles.*

In the above formula let

$$A + B = C, \quad A - B = D.$$

Then $$A = \frac{C+D}{2}, \quad B = \frac{C-D}{2},$$

and we have

$$\sin C = \sin\frac{C+D}{2}\cos\frac{C-D}{2} + \cos\frac{C+D}{2}\sin\frac{C-D}{2},$$

$$\sin D = \sin\frac{C+D}{2}\cos\frac{C-D}{2} - \cos\frac{C+D}{2}\sin\frac{C-D}{2},$$

$$\cos C = \cos\frac{C+D}{2}\cos\frac{C-D}{2} - \sin\frac{C+D}{2}\sin\frac{C-D}{2},$$

$$\cos D = \cos\frac{C+D}{2}\cos\frac{C-D}{2} + \sin\frac{C+D}{2}\sin\frac{C-D}{2}.$$

Adding and subtracting these, pair and pair,

$$\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2} \text{.....(5)},$$

$$\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2} \text{.....(6)},$$

$$\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2} \text{.....(7)},$$

$$\cos D - \cos C = 2\sin\frac{C+D}{2}\sin\frac{C-D}{2} \text{.....(8)}.$$

These should be remembered; some such way as the following is easy:

sum of two sines = twice sin(semi-sum). cos(semi-diff.),

diff. of two cosines = twice sin(semi-sum). sin(semi-diff.).

73. It will be noticed that we write $\cos D - \cos C$, although we wrote $\sin C - \sin D$. The reason is this: C is greater than D, and as the angle increases from 0° to 90°, the sine increases (see Arts. 44, 46), but the cosine diminishes; consequently, $\cos D$ is greater than $\cos C$, and we write $\cos D - \cos C$, so that the left-hand side of (8) may be positive; that is, we put the cosine of the smaller angle first.

74. These formulæ may be used inversely.

Thus if we have such an expression as

$$\sin\theta \cos 2\theta,$$

and we wish to express it as the difference of two sines.

By (5) we have

$$2\theta = \frac{C+D}{2}, \quad \theta = \frac{C-D}{2},$$

$$C + D = 4\theta, \quad C - D = 2\theta,$$

whence $$C = 3\theta, \quad D = \theta,$$

and by (5)

$$\sin\theta \,.\cos 2\theta = \tfrac{1}{2}(\sin 3\theta - \sin\theta),$$

so $$\sin 2\theta \,.\cos\theta = \tfrac{1}{2}(\sin 3\theta + \sin\theta),$$

$$\cos 2\theta \,.\cos\theta = \tfrac{1}{2}(\cos 3\theta + \cos\theta),$$

$$\sin\theta \,.\sin 2\theta = \tfrac{1}{2}(\cos\theta - \cos 3\theta).$$

75. *To find the tangent of the sum or difference of two angles in terms of the tangents of the separate angles, i.e. to find* $\tan(A \pm B)$ *in terms of* $\tan A$ *and* $\tan B$.

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$$

$$= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}.$$

Divide numerator and denominator by $\cos A \cos B$,

$$= \frac{\tan A + \tan B}{1 - \tan A \tan B} \quad \text{.....} (9),$$

so $$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}.$$

Since $\cot\theta = \dfrac{1}{\tan\theta}$, $\cot(A \pm B)$ is known.

76. If $B = 45°$. Then, since by Art. 28, $\tan 45° = 1$,

$$\tan(A + 45°) = \frac{\tan A + 1}{1 - \tan A},$$

$$\tan(A - 45°) = \frac{\tan A - 1}{1 + \tan A}.$$

77. *To express the sine, cosine, tangent, &c. of any angle in terms of the sine, cosine, &c. of half the angle.*

If in (1) we put $A = B$,

$$\sin 2A = 2 \sin A \cos A \quad \text{.....}(10),$$

and if $A = B$ in (3),

$$\cos 2A = \cos^2 A - \sin^2 A \quad \text{.....}(11)$$

$$= (1 - \sin^2 A) - \sin^2 A$$

$$= 1 - 2 \sin^2 A \quad \text{.....}(12),$$

or $$= \cos^2 A - (1 - \cos^2 A)$$

$$= 2 \cos^2 A - 1 \quad \text{.....}(13).$$

From (12) and (13),

$$1 - \cos 2A = 2 \sin^2 A,$$

$$1 + \cos 2A = 2 \cos^2 A,$$

whence $$\tan^2 A = \frac{1 - \cos 2A}{1 + \cos 2A}.$$

If in (9) we put $A = B$,

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A} \text{.....(14)},$$

$$\cot 2A = \frac{1-\tan^2 A}{2\tan A}.$$

Thus we have expressed the sine, cosine, tangent, &c. of an angle in terms of the ratios of half the angle.

Thus $\cos\theta$ may be written

$$\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2},\ 2\cos^2\frac{\theta}{2} - 1,\ \text{or}\ 1 - 2\sin^2\frac{\theta}{2},$$

for our $\theta = 2A$; therefore $\frac{\theta}{2} = A$.

Similarly $\sin 4\theta = 2\sin 2\theta \cos 2\theta$,

$$\tan 6\theta = \frac{2\tan 3\theta}{1-\tan^2 3\theta},$$

and so generally.

78. *To express the sine, cosine, &c. of an angle in terms of the sine, cosine, &c. of the third part of the angle.*

If in (1) we put $B = 2A$, we have

$$\begin{aligned}\sin 3A &= \sin A\cos 2A + \cos A\sin 2A\\ &= \sin A\,(1-2\sin^2 A) + \cos A\,(2\sin A\cos A)\ \text{by (12) (10)}\\ &= \sin A - 2\sin^3 A + 2\sin A\,(1-\sin^2 A)\\ &= 3\sin A - 4\sin^3 A \text{ ..... (15)}.\end{aligned}$$

From (3)

$$\begin{aligned}\cos 3A &= \cos 2A\cos A - \sin 2A\sin A\\ &= (2\cos^2 A - 1)\cos A - 2\sin A\cos A\,(\sin A)\ \text{by (13) (10)}\\ &= 2\cos^3 A - \cos A - 2\cos A\,(1-\cos^2 A)\\ &= 4\cos^3 A - 3\cos A \text{ .....(16)}.\end{aligned}$$

79. From the preceding,

$$\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2},$$

and $$1 = \cos^2 \frac{A}{2} + \sin^2 \frac{A}{2} \text{ (Art. 24)},$$

$$1 + \sin A = \left(\cos \frac{A}{2} + \sin \frac{A}{2}\right)^2,$$

$$1 - \sin A = \left(\cos \frac{A}{2} - \sin \frac{A}{2}\right)^2.$$

Taking the roots of both sides and adding,

$$2 \cos \frac{A}{2} = \sqrt{(1 + \sin A)} + \sqrt{(1 - \sin A)},$$

$$2 \sin \frac{A}{2} = \sqrt{(1 + \sin A)} - \sqrt{(1 - \sin A)}.$$

80. Thus if A be given we can find $\sin \frac{A}{2}$, $\cos \frac{A}{2}$, *provided that we know between what limits A lies.*

For a square root always supposes $\pm$ before it.

We have then a choice of four signs in the right hand of each of the above equations.

If we know between what limits A lies, we know within what limits $\frac{A}{2}$ lies; consequently, we know what sign the left hand should be, and can then effect this in the right hand by choosing the proper signs.

Suppose we are given that A lies between 450° and 630°, $\frac{A}{2}$ lies between 225° and 315°, *i.e.* either in the second or third quadrant.

Hence $\sin\frac{A}{2}$ is negative all the while, and $\sin\frac{A}{2}$ is numerically greater than $\cos\frac{A}{2}$ (see Table to Chap. VII.). Therefore $\sin\frac{A}{2} - \cos\frac{A}{2}$ is negative all the while,

and $\sin\frac{A}{2} + \cos\frac{A}{2}$ is also negative all the while,

(since each of these expressions will take their sign from that of $\sin\frac{A}{2}$). Hence

$$\sin\frac{A}{2} - \cos\frac{A}{2} = -\sqrt{(1 - \sin A)},$$

$$\sin\frac{A}{2} + \cos\frac{A}{2} = -\sqrt{(1 + \sin A)},$$

and $$2\sin\frac{A}{2} = -\sqrt{(1 + \sin A)} - \sqrt{(1 - \sin A)}.$$

81. *To find the sine, cosine, &c. of* 18°.

Let $$18° = \theta.$$

Then $$2\theta = 36°, \quad 3\theta = 54°.$$

Now $$2\theta + 3\theta = 36° + 54° = 90°,$$

or $$2\theta = 90° - 3\theta,$$

$$\sin 2\theta = \cos 3\theta,$$

$$2\sin\theta\cos\theta = 4\cos^3\theta - 3\cos\theta.$$

Dividing by $\cos\theta$,

$$\begin{aligned} 2\sin\theta &= 4\cos^2\theta - 3 \\ &= 4(1 - \sin^2\theta) - 3 \\ &= 1 - 4\sin^2\theta, \end{aligned}$$

$$4\sin^2\theta + 2\sin\theta = 1,$$

$$\sin^2\theta + \frac{1}{2}\sin\theta + \frac{1}{16} = \frac{1}{4} + \frac{1}{16} = \frac{5}{16},$$

or
$$\sin\theta = \frac{\pm\sqrt{(5)} - 1}{4}.$$

Now θ is in the first quadrant, and therefore $\sin\theta$ is positive. Hence we must take the positive sign, and

$$\sin 18^\circ = \frac{\sqrt{(5)} - 1}{4},$$

$$\cos 18^\circ = \sqrt{(1 - \sin^2 18^\circ)} = \frac{\sqrt{[\{10 + 2\sqrt{(5)}\}]}}{4},$$

$$\tan 18^\circ = \frac{\sqrt{(5)} - 1}{\sqrt{[\{10 + 2\sqrt{(5)}\}]}}.$$

Now
$$\sin 18^\circ = \cos 72^\circ,$$

and hence the trigonometrical ratios for 72° are known, also

$$\sin 54^\circ = \sin(3 \times 18^\circ),$$

and may be found, as also

$$\sin 36^\circ,\ \sin 9^\circ,\ \text{and}\ \sin 81^\circ.$$

We give some of the results:

$$\sin 9^\circ = \cos 81^\circ = \frac{\sqrt{[\{3 + \sqrt{(5)}\}]} - \sqrt{[\{5 - \sqrt{(5)}\}]}}{4},$$

$$\cos 9^\circ = \sin 81^\circ = \frac{\sqrt{[\{3 + \sqrt{(5)}\}]} + \sqrt{[\{5 - \sqrt{(5)}\}]}}{4},$$

$$\sin 36^\circ = \cos 54^\circ = \frac{\sqrt{[\{10 - 2\sqrt{(5)}\}]}}{4},$$

$$\cos 36^\circ = \sin 54^\circ = \frac{1 + \sqrt{(5)}}{4}.$$

82. A table is here given of the results that up to this point should be committed to memory.

$$D = G - \frac{G}{10}, \quad G = D + \frac{D}{9}, \quad \theta = \frac{\text{arc}}{\text{radius}},$$

π, circular measure of 180°,

$$\sin^2 A + \cos^2 A = 1, \quad \sec^2 A = 1 + \tan^2 A,$$

$$\sin^2 A = \frac{\tan^2 A}{1 + \tan^2 A}, \quad \cos^2 A = \frac{1}{1 + \tan^2 A},$$

$$\sin 0° = 0, \quad \sin 30° = \frac{1}{2}, \quad \sin 45° = \frac{1}{\sqrt{(2)}}, \quad \sin 60° = \frac{\sqrt{(3)}}{2}, \quad \sin 90° = 1,$$

$$\cos 0° = 1, \quad \cos 30° = \frac{\sqrt{(3)}}{2}, \quad \cos 45° = \frac{1}{\sqrt{(2)}}, \quad \cos 60° = \frac{1}{2}, \quad \cos 90° = 0,$$

$$\tan 0° = 0, \quad \tan 30° = \frac{1}{\sqrt{(3)}}, \quad \tan 45° = 1, \quad \tan 60° = \sqrt{(3)}, \quad \tan 90° = \infty.$$

General expression for given $\begin{cases} \text{sine} & \theta = n\pi + (-1)^n a, \\ \text{cosine} & \theta = 2n\pi \pm a, \\ \text{tangent} & \theta = n\pi + a; \end{cases}$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B,$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B;$$

$$\sin A + \sin B = 2 \sin \frac{A + B}{2} \cos \frac{A - B}{2},$$

$$\sin A - \sin B = 2 \cos \frac{A + B}{2} \sin \frac{A - B}{2},$$

$$\cos A + \cos B = 2 \cos \frac{A + B}{2} \cos \frac{A - B}{2},$$

$$\cos B - \cos A = 2 \sin \frac{A + B}{2} \sin \frac{A - B}{2};$$

$$\sin 2A = 2 \sin A \cos A,$$

$$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2 \sin^2 A = 2 \cos^2 A - 1,$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A},$$

$$\sin 3A = 3 \sin A - 4 \sin^3 A, \quad \cos 3A = 4 \cos^3 A - 3 \cos A.$$

CHAPTER X.

LOGARITHMS.

83. *Def.* The *logarithm* of a number to a given base is the index of the power to which the base must be raised in order to equal the given number.

Thus if $a^x = m$, x is the index to which a has to be raised in order to equal m. Hence x is the logarithm of m to the base a.

This is written $x = \log_a m$. The figure written below the line is the base, and we read the equation thus: log of m to base a is equal to x.

84. Suppose now the base is 10, *i.e.* $a = 10$, we know that 10^1 is 10, or, by our notation, $\log_{10} 10 = 1$; hence $\log_{10} 15$ will be 1 + a decimal.

We have then this definition:

The integral part of a logarithm is called its characteristic, the decimal part its mantissa.

Ex. $\log 20 = 1.30103$. Here 1 is the characteristic, .30103 the mantissa.

85. *To find the logarithm of unity.*

Since $a^0 = 1$, $\log_a 1 = 0$, and a being arbitrary, the *logarithm of unity to any base is zero.*

86. *To find the logarithm of a number with itself for its base.*

$a^1 = a$. Hence $\log_a a = 1$ *whatever a may be.*

87. *To shew that the logarithm of a product is equal to the sum of the logarithms of the factors, i. e.* to shew

$$\log_a(mn) = \log_a m + \log_a n.$$

Let $$\log_a m = x, \quad \log_a n = y,$$

$$m = a^x, \qquad n = a^y.$$

Hence $$mn = a^{x+y},$$

wherefore $$\log_a(mn) = x + y = \log_a m + \log_a n.$$

88. *To shew that the logarithm of a quotient is equal to the difference of the logarithms of the dividend and divisor, i. e.* to shew

$$\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n.$$

Let $$\log_a m = x, \quad \log_a n = y,$$

$$m = a^x, \qquad n = a^y,$$

$$\frac{m}{n} = \frac{a^x}{a^y} = a^{x-y},$$

whence $$\log_a\left(\frac{m}{n}\right) = x - y = \log_a m - \log_a n.$$

89. *To find the logarithm of infinity to any base.*

If a be an integer, $a^\infty = \infty$, and therefore $\log_a \infty = \infty$.

If a be a proper fraction, $\frac{1}{a^\infty} = \infty$, or $a^{-\infty} = \infty$, and $\log_a \infty = -\infty$.

90. *To find the logarithm of zero to any base.*

$$\log_a 0 = \log_a \frac{1}{\infty} = \log_a 1 - \log_a \infty = -\infty \text{ (by Art. 89)},$$

or, if a be a proper fraction,

$$\log_a 0 = \log_a \frac{1}{\infty} = \log_a 1 - \log_a \infty = \infty \text{ (by Art. 89)}.$$

91. *To shew that the logarithm of a number to any power is equal to the logarithm of the number, multiplied by the index of the power, i. e.*

$$\log_a(m^r) = r \log_a m.$$

Let $\log_a m = x, \quad a^x = m, \quad a^{xr} = m^r,$

whence $\log_a(m^r) = xr = r \log_a m.$

92. *To find a relation between the logarithms of the same number to different bases.*

Let $\log_a m = x, \quad \log_b m = y,$

$$a^x = m, \qquad b^y = m,$$

whence $a^x = b^y,$

$$a^{\frac{x}{y}} = b, \text{ or } \log_a b = \frac{x}{y} = \frac{\log_a m}{\log_b m},$$

$$b^{\frac{y}{x}} = a, \qquad \log_b a = \frac{y}{x} = \frac{\log_b m}{\log_a m}.$$

And we see that

$$\log_a m = \log_a b \log_b m,$$

$$\log_b m = \log_b a \log_a m,$$

and $\log_a b \times \log_b a = 1.$

93. There are two systems of logarithms in use. The first is called the *Napierian* system from its inventor. In this the base is 2.71828... which is called e. The other is the common system in which the base is 10. In treating of logarithms of the common system it is usual to omit the figure 10 below the line. Thus logarithm of eighteen in the common system is written $\log 18$.

This system has two great advantages which we will now explain.

94. *In the common system the characteristic may be determined by inspection.*

Since $10^0 = 1$, $10^1 = 10$, $10^2 = 100$, $10^3 = 1000\ldots$,

we have $0 = \log 1$, $1 = \log 10$, $2 = \log 100\ldots$.

Now any number between 1 and 10 lies between 10^0 and 10^1. Hence its logarithm (see Def.) must lie between 0 and 1, *i. e.* it will be a decimal.

So, again, any number between 100 and 1000, *i. e.* between 10^2 and 10^3 will have its logarithm between 2 and 3, *i. e.* 2 is the characteristic.

Let now N be a number, lying between 10^n and 10^{n+1}.

Then $\log N$ lies between n and $n + 1$, *i. e.* it is $n +$ a decimal. Hence n is the characteristic, *i. e.* the index of the highest power of n, which is less than the given number, is the characteristic.

Again, if N lie between $\frac{1}{10^n} \cdot \frac{1}{10^{n+1}}$, *i. e.* between 10^{-n}, $10^{-(n+1)}$, $\log N$ lies between $-n$ and $-(n + 1)$.

Thus it would be $-(n + \textit{decimal})$, *i. e.* $-n - \textit{decimal}$.

Now it is usual always to have the mantissa a positive quantity, hence we must take the higher number $-(n + 1)$ for our characteristic, and then the mantissa *must* be positive.

For $-n - \textit{decimal} = -n - 1 + (1 - \textit{decimal}) = -(n+1) + \textit{a decimal}$.

95. We see then that the characteristic may be determined by inspection by the following rules:

If the given number be greater than unity, the characteristic is the index of the power of 10 next below the number.

If the given number be less than unity, the characteristic is the index of the power of 10 next above the number, and is negative.

Thus the characteristic of 756 is 2, since it lies between 10^2 and 10^3.

That of .756 is -1, for it lies between $\frac{1}{10^1} \cdot \frac{1}{10^0}$, *i.e.* between $10^{-1} . 10^0$.

The above rules may be expressed more simply thus:

The characteristic of a number with integral figures is one less than the number of integral figures; if there are no integral figures, the characteristic is one more than the number of cyphers to the right of the decimal point, and is negative.

96. The second advantage of the common system is this:

If we know the logarithm of any number, we can at once write down the logarithm of the number when multiplied or divided by any power of 10; in other words, the mantissa will remain the same, the alteration taking place only in the characteristic.

Let N be a number whose logarithm is known,

$$\log_{10}(N \times 10^n) = \log_{10} N + \log_{10}(10^n) \text{ (Art. 87)}$$
$$= \log_{10} N + n \log_{10} 10 \text{ (Art. 91)}$$
$$= \log N + n \text{ (Art. 86).}$$

So $\quad \log_{10}(N \div 10^n) = \log N - n.$

So that if the original number be multiplied by any multiple of 10, we have to *add* the index of that multiple to the known logarithm; if divided, we have to subtract that index, always taking care that the mantissa shall be positive.

In other words, *if we write down any number whose logarithm is known, then we may add or prefix any number of cyphers, or mark off any number of decimal places, and the mantissa of the logarithm of the new number will remain the same as that of the original number, but the characteristic will be increased or diminished by n, as the case may be.*

Thus $\log 405 = 2.607455,$

$$\log 40500 = \log(405 \times 10^2) = 2 + 2.607455 \text{ by the above}$$
$$= 4.607455,$$

$$\log 4.05 = \log \frac{405}{100} = 2.607455 - 2 = .607455,$$

$$\log .0405 = \log \frac{4.05}{10^2} = -2 + .607455,$$

or, as it is written, $= \bar{2}.607455.$

97. For the construction and use of logarithmic tables, the reader is referred to Todhunter's *Plane Trigonometry*, Chap. XXI., being too long and difficult an explanation to be introduced here.

98. When we have an equation we may take logarithms of both sides to any base. Thus if $\frac{\sin A}{a} = \frac{\sin B}{b}$, and we wish to take logarithms to the base 10, we shall have

$$\log_{10} \frac{\sin A}{a} = \log_{10} \frac{\sin B}{b},$$

or, by previous rules, (Arts. 88, 93)

$$\log \sin A - \log a = \log \sin B - \log b.$$

99. The tabulated logarithms of sines, cosines, &c. are always increased by 10, and such logarithm is denoted by L, *i.e.*

$$\log \sin A + 10 = L \sin A.$$

These are called tabulated logarithms, and are always used.

The equation of Art. 98 will thus become

$$L \sin A - 10 - \log a = L \sin B - 10 - \log b,$$

or $$L \sin A - \log a = L \sin B - \log b.$$

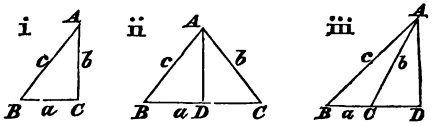
In any case of course when we have taken the logarithms of both sides, the rules still apply.

Thus $a^x = y$, $\log(a^x) = \log y$, or $x \log a = \log y$.

CHAPTER XI.

RELATIONS BETWEEN THE SIDES AND ANGLES OF TRIANGLES.

100. Let ABC be a triangle, A, B, C the angles, the sides are usually denoted by the small letters corresponding to the angle opposite to them.



Thus $\qquad BC = a, \; AC = b, \; AB = c.$

101. *To shew that in any triangle*

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c},$$

i.e. each side is proportional to the side of the opposite angle.

Draw AD at right angles to BC, or BC produced.

Then, in fig. ii, where C is an acute angle,

$$AD = AB \sin B, \quad AD = AC \sin C,$$

$$AB \sin B = AC \sin C,$$

$$c \sin B = b \sin C,$$

or

$$\frac{\sin B}{b} = \frac{\sin C}{c}.$$

In fig. iii, C is an obtuse angle,

$$AD = AC \sin ACD = AC \sin(180° - C)$$
$$= AC \sin C,$$

$$AD = AB \sin B,$$

whence $$b \sin C = c \sin B,$$

and $$\frac{\sin B}{b} = \frac{\sin C}{c}.$$

In fig. i, where C is a right angle, C and D coincide, and we have

$$\sin B = \frac{b}{c} = \frac{\sin B}{\sin C}, \text{ for } \sin 90° = 1 \text{ (Art. 45)},$$

the same result as before.

Similarly, by dropping perpendiculars from B on the opposite side, we may shew that in every case

$$\frac{\sin C}{c} = \frac{\sin A}{a}.$$

Hence we have always

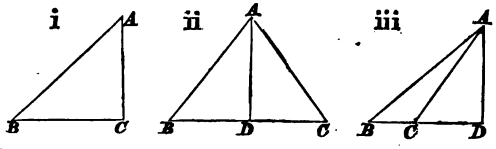
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

We shall find hereafter (Art. 121) that each of these ratios is equal to $\frac{1}{2R}$, where R is the radius of the circle circumscribing the triangle.

102. In a right-angled triangle we can express any of the sides in terms of a side and some ratio of one of the acute angles, by Art. 23.

103. *To find an expression for the cosine of an angle in terms of the sides.*

Let ABC be a triangle; to find $\cos B$.



Draw AD at right angles to BC, or BC produced if necessary.

From fig. ii and Euc. II. 13,

$$AC^2 = AB^2 + BC^2 - 2CB \,.\, BD$$
$$= AB^2 + BC^2 - 2CB \,.\, AB \cos B.$$

From fig. iii and Euc. II. 12,

$$AC^2 = AB^2 + BC^2 + 2CB \,.\, BD$$
$$= AB^2 + BC^2 + 2CB \,.\, AB \cos(180^\circ - C)$$
$$= AB^2 + BC^2 - 2CB \,.\, AB \cos C.$$

In fig. i, D and C coincide, and we have

$$AC^2 = AB^2 - BC^2$$
$$= AB^2 + BC^2 - 2BC^2$$
$$= AB^2 + BC^2 - 2BC \,.\, AB \cos B.$$

Hence in any case

$$b^2 = a^2 + c^2 - 2ac \cos B,$$

and $$\cos B = \frac{a^2 + c^2 - b^2}{2ac},$$

so $$\cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

The symmetrical character of these formulæ will assist the memory.

104. *To express* $\sin\frac{B}{2}$, $\cos\frac{B}{2}$, *&c. in terms of the sides.*

By the last article

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}.$$

By Art. 77, $\quad 1 + \cos B = 2\cos^2\frac{B}{2}.$

Now
$$1 + \cos B = 1 + \frac{a^2 + c^2 - b^2}{2ac}$$
$$= \frac{2ac + a^2 + c^2 - b^2}{2ac}$$
$$= \frac{(a + c)^2 - b^2}{2ac}$$
$$= \frac{(a + c + b)(a + c - b)}{2ac}.$$

Let $2s$ = sum of sides $= a + b + c$,

$$2(s - b) = a + b + c - 2b = a + c - b.$$

Whence $\quad 1 + \cos B = \frac{2s \cdot 2(s - b)}{2ac} = \frac{2 \cdot s(s - b)}{ac},$

or $\quad 2\cos^2\frac{B}{2} = 2 \cdot \frac{s(s - b)}{ac},$

$$\cos\frac{B}{2} = +\sqrt{\left\{\frac{s(s - b)}{ac}\right\}}.$$

The positive sign is used because since $A + B + C = 180°$ always; therefore A, B, C are each less than 180°, and $\frac{A}{2}$, $\frac{B}{2}$, $\frac{C}{2}$ less than 90°, and their sines, cosines, &c. positive.

Again, $2\sin^2\frac{B}{2} = 1 - \cos B$ (by Art. 77)

$$= 1 - \frac{a^2 + c^2 - b^2}{2ac}$$

$$= \frac{2ac - a^2 - c^2 + b^2}{2ac}$$

$$= \frac{b^2 - (a-c)^2}{2ac}$$

$$= \frac{\{b - (a-c)\}\{b + (a-c)\}}{2ac}$$

$$= \frac{(b + c - a)(a + b - c)}{2ac}$$

$$= \frac{2(s-a)\,2(s-c)}{2ac} \text{ (see former part),}$$

or $$\sin^2\frac{B}{2} = \frac{(s-a)(s-c)}{ac},$$

and $$\sin\frac{B}{2} = +\sqrt{\left\{\frac{(s-a)(s-c)}{ac}\right\}}.$$

Hence $$\tan\frac{B}{2} = \frac{\sin\frac{B}{2}}{\cos\frac{B}{2}} = \sqrt{\left\{\frac{(s-a)(s-c)}{s(s-b)}\right\}},$$

so $$\sin\frac{A}{2} = \sqrt{\left\{\frac{(s-b)(s-c)}{bc}\right\}},$$

$$\cos\frac{A}{2} = \sqrt{\left\{\frac{s(s-a)}{bc}\right\}},$$

$$\tan\frac{A}{2} = \sqrt{\left\{\frac{(s-b)(s-c)}{s(s-a)}\right\}};$$

$$\sin\frac{C}{2} = \sqrt{\left\{\frac{(s-a)(s-b)}{ab}\right\}},$$

$$\cos\frac{C}{2} = \sqrt{\left\{\frac{s(s-c)}{ab}\right\}},$$

$$\tan\frac{C}{2} = \sqrt{\left\{\frac{(s-a)(s-b)}{s(s-c)}\right\}},$$

where a certain symmetry may be observed.

105. Since $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$ (Art. 77),

$$\sin A = \frac{2}{bc}\sqrt{\{s(s-a)(s-b)(s-c)\}},$$

which will be useful hereafter.

CHAPTER XII.

SOLUTION OF RIGHT-ANGLED TRIANGLES.

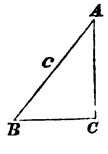
106. There are six *elements* in every triangle, the three angles and the three sides. Of these, if we know *three* we can determine the rest, unless they be the three angles, when we can only determine the *ratio* of the sides, since there may be drawn any number of triangles *equiangular* to a given triangle, whose sides are of different length to the original triangle.

It will be well to refer to Arts. 98, 99 at the end of the chapter on logarithms, for their use in what follows here.

The system here used will be that to the base 10, and our object is, *to find the remaining elements by the aid of those given us.* We shall first take right-angled triangles, C is supposed to be the right angle.

107. *To solve a right-angled triangle, having given the hypothenuse and an acute angle.*

Given $C = 90°$, c, B. To find A, a, b.



$$A = 90° - B,$$

$$a = c \cos B \text{ (Art. 23) and } a \text{ is known.}$$

Also $$a = c \sin A \text{ (Art. 18),}$$

$$\log a = \log c + L \sin A - 10 \text{ (Arts. 98, 99);}$$

$$b = c \sin B,$$

$$\log b = \log c + L \sin B - 10,$$

$L \sin B$ is found from the tables, since B is given.

Hence A, a, b are determined.

108. *To solve a right-angled triangle, having given the hypothenuse and a side.*

Given $C = 90°$, c, a. To find A, B, b.

$$\sin A = \frac{a}{c}, \quad L \sin A - 10 = \log a - \log c,$$

and $$L \sin A = \log a - \log c + 10.$$

This gives A, and $B = 90° - A$, and is therefore known.

Also $$b = c \cos A,$$

$$\log b = \log c + L \cos A - 10.$$

Thus A, B, and b are known.

109. *To solve a right-angled triangle, having given a side and an acute angle.*

Given $C = 90°$, a, A. To find B, c, b.

$B = 90° - A$ and is known.

$$\frac{a}{c} = \sin A, \quad \log a - \log c = L \sin A - 10,$$

$$\log c = \log a - L \sin A + 10,$$

$$\frac{a}{b} = \tan A,$$

$$\log b = \log a - L \tan A + 10.$$

Thus B, c, b are determined.

If a and B are given,

$$A = 90° - B,$$

A is known, and then c, b are found by a similar method to the above.

110. *To solve a right-angled triangle, having given the two sides.*

Given $C = 90°$, a, b. To find A, B, c.

$$\frac{a}{b} = \tan A,$$

$$L \tan A = \log a - \log b + 10, \ A \text{ is known},$$

$$B = 90° - A \text{ and is known};$$

$$\frac{a}{c} = \sin A,$$

$$\log c = \log a - L \sin A + 10.$$

Thus c is known.

CHAPTER XIII.

TO SOLVE OBLIQUE-ANGLED TRIANGLES.

111. *To solve a triangle, having given two angles and a side.*

Given A, C, b. To find B, a, c.

$$B = 180° - A - C \text{ and is known,}$$

$$\frac{c}{\sin C} = \frac{b}{\sin B}, \quad \log c - L \sin C = \log b - L \sin B \text{ (Art. 99),}$$

$$\log c = \log b + L \sin C - L \sin B,$$

so also $$\log a = \log b + L \sin A - L \sin B.$$

Thus B, a, c are determined.

112. *To solve a triangle, having given two sides and the included angle.*

Given b, c, A. To find B, C, a.

$$\frac{\sin B}{\sin C} = \frac{b}{c},$$

$$\frac{\sin B - \sin C}{\sin B + \sin C} = \frac{b - c}{b + c},$$

Todhunter's *Algebra for Beginners* (Art. 367),

or $$\frac{2 \cos \frac{B + C}{2} \sin \frac{B - C}{2}}{2 \sin \frac{B + C}{2} \cos \frac{B - C}{2}} = \frac{\tan \frac{1}{2}(B - C)}{\tan \frac{1}{2}(B + C)} = \frac{b - c}{b + c} \text{ (Art. 72).}$$

Now $$\frac{B + C}{2} = \frac{180° - A}{2} = 90° - \frac{A}{2} \text{ and is known.}$$

Wherefore $\tan\frac{1}{2}(B+C) = \tan\left(90^\circ - \frac{A}{2}\right) = \cot\frac{A}{2}$ is known.

Hence $L\tan\frac{1}{2}(B-C) = \log(b-c) - \log(b+c) + L\cot\frac{A}{2}$.

This gives $\frac{1}{2}(B-C)$ and $\frac{1}{2}(B+C)$ is known;

therefore B and C are known.

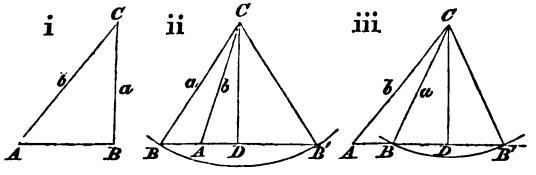
$$\frac{a}{c} = \frac{\sin A}{\sin C},$$

$$\log a = \log c + L\sin A - L\sin C,$$

which gives a.

113. *To solve a triangle, having given two sides and an angle opposite to one.*

Given a, b, A. To find B, C, c.



The three figures above give every variety of triangles that can occur with the given conditions, viz.

$$CB = a, \quad CA = b, \quad CAB = A.$$

CD is a perpendicular upon c (produced if necessary).

BB' is a circle with centre C and radius CB cutting AD produced in B'. Join CB'.

In fig. i CB and CD coincide or $CB = CD$.

In figs. ii and iii CD is always less than CB.

Hence CD is never greater than CB, in other words, $b\sin A$ is never greater than a, or $\frac{b}{a}\sin A$ never greater than unity.

(α) If then our given parts be such as to make $\frac{b}{a}\sin A$ greater than unity, the triangle is impossible.

Our first condition is therefore $\frac{b}{a}\sin A$, *must not be greater than unity.*

(β) In fig. i $a = b\sin A$ or $\frac{b}{a}\sin A = 1$.

Hence when $\frac{b}{a}\sin A$ is unity the triangle is right-angled at B.

(γ) In fig. ii CAB is evidently a triangle satisfying the given conditions.

Now the triangle CAB' has $CA = b$, $CB' = a$, but the angle $CAB' = 180° - CAB = 180° - A$, and therefore does not satisfy the conditions.

We observe in this case that b is less than a, B an acute angle, and $\frac{CD}{CB}$, *i. e.* $\frac{b}{a}\sin A$ less than unity.

Hence, when $\frac{b}{a}\sin A$ is less than unity, and b less than a, there is only *one* triangle with the given parts, and that one is acute angled at B.

(δ) In fig. iii CAB satisfies the given conditions. But we see that CAB' does also, for $CA = b$, $CB' = a$, and CAB is the same angle as CAB'.

Hence there are two triangles in this case satisfying the given conditions.

We observe in this case that b is greater than a, $\frac{b}{a}\sin A$ less than unity, and B either acute or obtuse.

Hence, if $\frac{b}{a}\sin A$ be less than unity, and b greater than a, there are two triangles.

This is called the *ambiguous case*, for we are uncertain which triangle to take.

From the above we gather that when two sides and the angle opposite to one are given, there may be (α) *no* triangle, (β) (γ) *one* triangle, or (δ) *two* triangles.

114. We have hitherto deduced our results from geometrical considerations; we proceed to find the same by the aid of Trigonometry.

We are given a, b, A. To find B, we have

$$\frac{\sin B}{b} = \frac{\sin A}{a}, \text{ or } \sin B = \frac{b}{a}\sin A.$$

(α') If now $b \sin A$ be greater than a, *i. e.* $\frac{b \sin A}{a}$ greater than unity, $\sin B$ is greater than unity, which (Art. 53) is impossible. Hence if $\frac{b}{a}\sin A$ is greater than unity, no triangle exists.

(β') If $\frac{b}{a}\sin A = 1$, $b \sin A = a$ and B is $90°$ (Art. 23).

(γ') If $\frac{b}{a}\sin A$ be less than unity and b less than a, we must have B less than A (Euc. I. 18), and $\sin B$ less than $\sin A$.

But since A and B are less than $180°$, and B less than A, B must be less than $90°$, *i. e.* acute.

There is therefore *one* triangle in this case.

(δ') If $\frac{b}{a}\sin A$ be less than unity, and b greater than a, we must have B greater than A, and $\sin B$ greater than $\sin A$. In this case there is no such restriction of B, for $\sin B = \sin(180° - B)$, and B may lie on either side of $90°$

without violating the condition that $\sin B$ must be greater than $\sin A$. Hence there may be two values of B that will satisfy the conditions, one greater than 90°, the other less; which we saw in the last article was called the *ambiguous case.*

The last two articles have been arranged so as to be easily compared. The results are evidently the same in both.

115. *To solve the triangle, having given all the sides.*

Let $$2s = a + b + c,$$

$$\tan\frac{A}{2} = \sqrt{\left\{\frac{(s-b)(s-c)}{s(s-a)}\right\}} \text{ (Art. 104).}$$

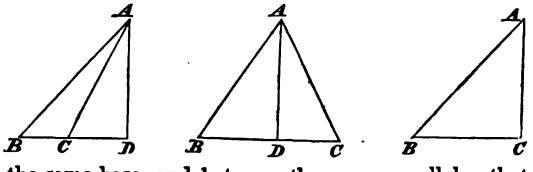
Similarly B, C may be found.

CHAPTER XIV.

PROPERTIES OF TRIANGLES.

116. *To find the area of a triangle.*

The area of a triangle is, by Euclid, half the rectangle



on the same base, and between the same parallels; that is, if S be the area of the triangle,

$$S = \frac{1}{2} BC.AD = \frac{1}{2} BC.AC \sin C$$

$$= \frac{1}{2} ab \sin C \text{.....(i)};$$

that is, *half the product of two sides into the sine of the included angle.*

Now, by Art. 105,

$$\sin C = \frac{2}{ab} \sqrt{\{s(s-a)(s-b)(s-c)\}}.$$

Hence $$S = \sqrt{\{s(s-a)(s-b)(s-c)\}} \text{ ..... (ii).}$$

Also $$a = b \frac{\sin A}{\sin B}.$$

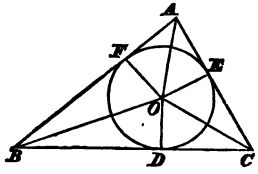
Hence, by (i), $$S = \frac{1}{2} b^2 \frac{\sin A \sin C}{\sin B} \text{ ..... (iii)};$$

that is, we can find the area of a triangle if a side and two angles are known, for then the third angle is known.

Results (i) and (ii) are the most useful.

117. *To find the radius of the circle inscribed in a triangle.*

ABC is the given triangle. Construct as in Euc. IV. 4.



Let FED be the circle. $OD = OF = OE = r$.

Then $S =$ area $AOB +$ area $BOC +$ area COA

$$= \frac{1}{2} AB.r + \frac{1}{2} BC.r + \frac{1}{2} AC.r$$

$$= r\left(\frac{c + a + b}{2}\right) = rs \text{ (Art. 104).}$$

Hence $\quad r = \dfrac{S}{s}$.

118. By Euc. III. 36. Since BD, BF are tangents to the circle from B,

$$BF = BD, \text{ so } CE = CD, \text{ and } AF = FE.$$

Hence $\quad 2(BD + DC + AE) = a + b + c = 2s,$

$$a + AE = s,$$

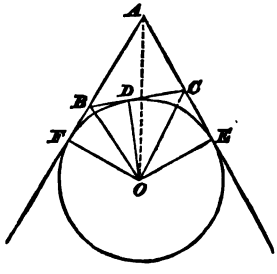
$$AE = s - a = AF,$$

so

$$BD = s - b = BF,$$

$$CD = s - c = CE.$$

119. *To find the radius of the escribed circle, i. e.* of the circle which touches one side of a triangle, and the other two produced.



Let FDE be such a circle touching BC, *i. e.* a in D and the other sides produced in E, F. O its centre. By the geometry OB, OC bisect the angles at B and C.

Let $$OF = OD = OE = r_a,$$
being the radius of the circle escribed opposite to the angle A.

The figure $ABOC$ = area ABO + area ACO

$$= \frac{1}{2}\{cr_a + br_a\}$$
$$= r_a \frac{c+b}{2}.$$

But figure $ABOC$ = area ABC + area OBC

$$= S + \frac{1}{2} ar_a,$$

whence $$r_a \frac{c+b}{2} = S + \frac{1}{2} ar_a,$$

$$r_a = \frac{S}{\frac{c+b-a}{2}} = \frac{S}{s-a},$$

so $$r_b = \frac{S}{s-b}, \quad r_c = \frac{S}{s-c}.$$

These and the radius of the inscribed circle $r = \frac{S}{s}$ are remarkably symmetrical.

120. Since $BD = BF$ (Euc. III. 36) being both tangents from B, and $CD = CE$ for a similar reason,

$$AF + AE = AB + AC + (BD + DC) = 2s.$$

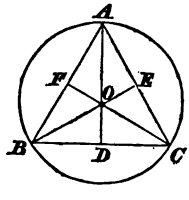
But $\qquad AF = AE$ (Euc. III. 36).

Hence $\qquad AF = AE = s$ and $AB + BD = s$.

From this we see that if two tangents be drawn to a circle, and a third tangent be drawn in any position so as to form a triangle with the other two tangents, the sum of the sides of all such triangles will be invariable, since it will always be twice either of the first two tangents. The line AE is therefore that quantity which we have denoted by s, that is $\frac{a+b+c}{2}$.

121. *To find the radius of the circle circumscribed about a triangle.*

Let O be the centre. Draw OD, OE, OF at right angles to the sides. These bisect the sides, Euc. IV. 5.



Let $\qquad R = OB = OC = OA.$

The angle BOC = twice the angle $BAC = 2A$ (Euc. III. 20).

Hence $\qquad BOD = A,$

and $\qquad BD = OB \sin BOD,$

or $\qquad \frac{a}{2} = R \sin A,$

$$R = \frac{a}{2\sin A} = \frac{a}{2\frac{2}{bc}\sqrt{\{s(s-a)(s-b)(s-c)\}}}$$

$$= \frac{abc}{4S} \text{ (by Art. 116)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}.$$

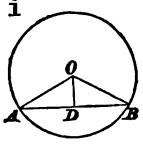
Hence we have

$$\frac{\sin A}{a} = \frac{1}{2R} = \frac{\sin B}{b} = \frac{\sin C}{c} \text{ (by Art. 101)}.$$

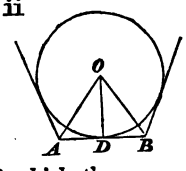
122. *To find the radii of the inscribed and circumscribed circles of a regular polygon of n sides.*

Let AB be the side of a regular polygon of n sides, O the centre of the circumscribed circle, OA its radius $= R$. Let $AB = a$, fig. i.

i



ii



Now the angles at O, of which there are n equal to AOB, together $= 360° = 2\pi$, Euc. I. 32.

Therefore $n \times AOB = 2\pi$, $AOB = \frac{2\pi}{n}$, $AOD = \frac{\pi}{n}$,

$$AD = AO \sin AOD, \text{ or } \frac{a}{2} = R\sin\frac{\pi}{n}, \quad R = \frac{a}{2\sin\frac{\pi}{n}} \ldots \text{(i)}.$$

If O be the centre of the inscribed circle, $OD = r$ its radius, fig. ii, as before $AOD = \frac{\pi}{n}$,

$$OD = AD\cot\frac{\pi}{n}, \quad r = \frac{a}{2}\cot\frac{\pi}{n} = \frac{a}{2\tan\frac{\pi}{n}} \ldots \text{(ii)}.$$

123. *To find the areas of the polygons in terms of R, r.*

Area of polygon, in fig. i,

$$= n \times AOB = n \times \frac{1}{2} AB.OD$$

$$= n \frac{a}{2} \cdot \frac{a}{2} \cot \frac{\pi}{n} = \frac{na^2}{4} \cot \frac{\pi}{n}$$

$$= nR^2 \sin^2 \frac{\pi}{n} \cot \frac{\pi}{n}, \text{ Art. 22, (i),}$$

$$= nR^2 \sin \frac{\pi}{n} \cos \frac{\pi}{n} \text{ ..... (i).}$$

Also area, by fig. ii,

$$= n \times AOB = n \times \frac{1}{2} a.OD$$

$$= n \times AD.OD$$

$$= n \frac{a}{2} \cdot \frac{a}{2} \cot \frac{\pi}{n}$$

$$= n \frac{a^2}{4} \cot \frac{\pi}{n}$$

$$= nr^2 \tan^2 \frac{\pi}{n} \cot \frac{\pi}{n}, \text{ by Art. 122 (ii),}$$

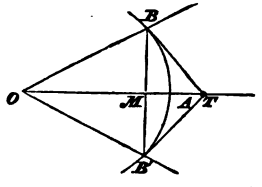
$$= nr^2 \tan \frac{\pi}{n} \text{ .....(ii).}$$

CHAPTER XV.

ON THE CIRCLE.

124. *If θ be the circular measure of an angle less than a right angle, θ is greater than* $\sin\theta$ *and less than* $\tan\theta$. *And the limit of* $\frac{\sin\theta}{\theta}$ *and* $\frac{\tan\theta}{\theta}$, *when θ is indefinitely decreased, is unity.*

Let AOB be an angle less than a right angle, and θ its circular measure.



Make on the other side of AO the angle $AOB' = AOB$.

With centre O and any radius OA describe the circle BAB'.

Draw BT a tangent at B, meeting OA produced in T.

Join $B'T$. Then since $OB, OT = OB', OT$ each to each and contain equal angles; TB' is also a tangent at B', and $TB = TB'$. Draw BMB', cutting OT at right angles, as it evidently will.

We assume, as an axiom, that the arc BAB' is greater than BMB', and less than $BT + TB'$.

Hence BA being half BAB' is greater than BM and less than BT; that is, $\frac{BA}{OB}$ is greater than $\frac{BM}{OB}$ and less than $\frac{BT}{OB}$, or θ is greater than $\sin\theta$ and less than $\tan\theta$.

Therefore $\sin\theta$, θ, $\tan\theta$ are in ascending order of magnitude.

Hence 1, $\dfrac{\theta}{\sin\theta}$, $\dfrac{1}{\cos\theta}$ are in ascending order of magnitude. Now when θ is indefinitely diminished $\cos\theta = 1$ (Art. 46).

Consequently, since $\dfrac{\theta}{\sin\theta}$ lies between 1 and $\dfrac{1}{\cos\theta}$, *i.e.* between 1 and 1, the limit to which $\dfrac{\theta}{\sin\theta}$ tends is unity.

Hence $\dfrac{\sin\theta}{\theta}$ becomes unity when θ is very small and also

$$\frac{\sin\theta}{\theta} \times \frac{1}{\cos\theta} \text{ or } \frac{\tan\theta}{\theta}.$$

125. *To find the area of a circle, radius r.*

The area of a regular polygon of n sides described about a circle, radius r is, by Art. 123, $= nr^2 \tan\dfrac{\pi}{n}$.

Now as n is indefinitely increased, the area of the polygon approaches nearer and nearer to that of the circle.

Hence $$\text{area of circle} = \text{limit } nr^2 \tan\frac{\pi}{n}$$

$$= \text{limit } \pi r^2 \left(\frac{\tan\dfrac{\pi}{n}}{\dfrac{\pi}{n}} \right).$$

And when n is very large, $\dfrac{\pi}{n}$ is very small, and therefore, by Art. 124,

$$\text{limit of } \frac{\tan\dfrac{\pi}{n}}{\dfrac{\pi}{n}} = 1.$$

Hence $$\text{area of circle} = \pi r^2.$$

126. *To find the area of a sector.*

Let θ be the circular measure of the angle of the sector. Then

$$\frac{\text{area of sector}}{\text{area of circle}} = \frac{\theta}{2\pi}.$$

Hence area of sector $= \dfrac{\theta}{2\pi} \times$ (area of circle)

$$= \frac{\theta}{2\pi} \cdot \pi r^2, \text{ by the last article,}$$

$$= \frac{r^2\theta}{2}.$$

Now $\theta = \dfrac{\text{arc of sector}}{r}$ (Art. 10), or arc of sector $= r\theta$.

Hence the area of a sector = half the product of the length of the arc into the radius.

127. Subjoined is a table of those results obtained in the last chapters which it is useful to remember.

$$\log_a 1 = 0, \quad \log_a a = 1, \quad \log_a(mn) = \log_a m + \log_a n,$$

$$\log_a \frac{m}{n} = \log_a m - \log_a n, \quad \log_a(m^r) = r \log_a m.$$

$$L \sin A = \log \sin A + 10.$$

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{1}{2R}.$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}, \quad \sin C = \frac{2}{ab}\sqrt{\{s(s-a)(s-b)(s-c)\}},$$

$$\sin\frac{C}{2} = \sqrt{\left\{\frac{(s-a)(s-b)}{ab}\right\}}, \quad \cos\frac{C}{2} = \sqrt{\left\{\frac{s(s-c)}{ab}\right\}}.$$

$$\text{Area of triangle} = \frac{1}{2}ac\sin B = \sqrt{\{s(s-a)(s-b)(s-c)\}}.$$

$$\text{Radius of inscribed circle, } r = \frac{S}{s}.$$

$$\text{Radius of escribed circle, } r_a = \frac{S}{s-a}.$$

$$\text{Radius of circumscribed circle, } R = \frac{abc}{4S}.$$

$$\text{Area of polygon about a circle} = nr^2 \tan\frac{\pi}{n}.$$

$$\text{Area of circle} = \pi r^2.$$

$$\text{Area of sector} = \frac{r^2\theta}{2}.$$

EXAMPLES.

I. Arts. 2—6.

1. Express the angle 15° 4′ 17″ in the centesimal method.

By our formula, Art. 6,

$$G = D + \frac{D}{9}.$$

We must first bring 15° 4′ 17″ to the decimal of a degree. By the usual arithmetical process, we have

$$\begin{array}{l} 6{,}0)\ \underline{17.} \\ 6{,}0)\ \underline{4.28\dot{3}} \\ \quad 15.0713\dot{8} \end{array}$$

$$D = 15^\circ.0713\dot{8},$$

$$\frac{D}{9} = 1.67459.$$

Hence adding, we get

$$G = 16.74597 = 16^g\ 74^{\backprime}\ 59^{\backprime\backprime}.7 \text{ nearly (Art. 5).}$$

2. Express (1) 113° 0′ 35″, (2) 78° 50′ 41″.3,
(3) 56° 16′ 42″, (4) 0° 36′ 15″,

in grades, minutes, &c.

3. Express $73^g\ 81^{\backprime}\ 5^{\backprime\backprime}$ in the English mode.

Our formula gives

$$D = G - \frac{G}{10},$$

$$G = 73.8105 \text{ (Art. 5)},$$

$$\frac{G}{10} = \begin{array}{r} 7.38105 \\ \hline 66.42945 \end{array} \text{ by subtracting}$$

$$\begin{array}{r} 60 \\ \hline 25.76700 \\ 60 \\ \hline 46.020 \end{array}$$

And $$D = 66^\circ\ 25'\ 46''.$$

4. Express (1) $125^g\ 56'\ 63''.58$, (2) $87^g\ 60'\ 53''.39$, (3) $62^g\ 53'\ 15''$, (4) $0^g\ 60'\ 4''.16$, in the English method.

5. The sum of two angles is 80 grades, their difference is $\frac{2}{5}$ of a right angle. Find the angles in degrees.

6. If 18° is the unit, what will represent 81°?

7. Find the number of grades in the unit of angular measure when an angle of 135° is represented by 15.

If 135° is represented by 15, 9° is represented by 1, *i.e.* 10 is the number of grades in the unit of angular measurement.

8. How many degrees are there in the unit of angular measure when $11\frac{1}{4}°$ is represented by $\frac{1}{8}$?

9. Divide a right angle into two parts such that the ratio of them may be that of 11 : 7.

II. Arts. 7—15.

1. Find the circular measure of 9°.

We have, Art. 8, since π is the circular measure of 180°.

$$\frac{\theta}{\pi} = \frac{9^\circ}{180^\circ} = \frac{1}{20}, \text{ whence } \theta = \frac{\pi}{20}.$$

2. Find the circular measure of

(1) 57°, (2) 57° 30′, (3) 18°, (4) 22° 30′, (5) 225°.

3. Find the number of degrees, &c. in an angle whose circular measure is $2\frac{1}{2}$.

As in Art. 13, $\frac{x}{180^\circ} = \frac{\theta}{\pi}$, here $\theta = \frac{5}{2}$,

$$x = \frac{5}{2} \cdot \frac{180^\circ}{\pi} = \frac{5}{2} \cdot \frac{180^\circ}{3.14159} = \frac{5}{2} \times 57^\circ.2957 = 143^\circ.2392,$$

which will give $x = 143^\circ\ 14'\ 21''.3$.

4. Find the angles whose circular measure are

$$1,\ \frac{1}{2},\ \frac{3}{2},\ \frac{1}{3},\ \frac{2}{3},\ \frac{5}{6}.$$

5. Find the number of grades, &c. in the angles whose circular measure are those in (4).

6. Find the number of degrees in an angle subtended at the centre by an arc 5 feet in length, the radius being 10 yards.

7. The angles of a triangle are in Arithmetical Progression; the circular measure of the difference between the first and second is $\frac{7\pi}{60}$. Find them.

(The three angles of any triangle = 180°, Euc. I. 32).

8. The angles of a triangle are in Geometrical Progression; the circular measure of the smallest is to that of 90° as $1 : 3\frac{1}{2}$. Find them.

III.

1. Find the complement of

$$39^\circ\ 15'\ 28'',\quad 74^\circ\ 16'\ 3'',$$

and the supplement of

$$16^\circ\ 3'\ 27'',\quad 176^\circ\ 19'\ 21'',$$ see Arts. 18, 19.

2. Write down in trigonometrical notation the product of the sine of A into the cosine of B into the square root of the product of the squares of the secant of A and the cosecant of B (Arts. 16, 21, 22). Write the result in its simplest form.

3. Simplify the following and express it as the product of two factors:

$$\sin^2\theta + \sin\theta\,(\cos\theta + \tan\theta) - \cos\theta\,(\sin\theta + \tan\theta)$$
$$- \tan\theta\,(\sin\theta + \cos\theta) - \cos^2\theta - \tan^2\theta.$$

4. If $\sin^2 a \cos^2\beta = \dfrac{\cot^2\beta}{3\operatorname{cosec}^2 a}$, what is the value of $\sin\beta$?

5. ABC is a triangle, AD is a perpendicular on BC, the angle $ABC = \phi$, $ACB = \theta$, shew that

$$BD \tan\phi = CD \tan\theta,$$

and $$AB \operatorname{cosec}\theta = AC \operatorname{cosec}\phi \text{ (Art. 23).}$$

IV. By the aid of Arts. 24, 25.

1. Shew $\sin^2 A - \cos^2 B = \sin^2 B - \cos^2 A$.

Taking the left-hand side, we have

$$\begin{aligned}\sin^2 A - \cos^2 B &= (1 - \cos^2 A) - (1 - \sin^2 B)\\ &= 1 - \cos^2 A - 1 + \sin^2 B\\ &= \sin^2 B - \cos^2 A.\end{aligned}$$

2. $\dfrac{1}{\sin\theta\cos\theta} = \tan\theta + \cot\theta$,

for (Art. 24) $\dfrac{1}{\sin\theta\cos\theta} = \dfrac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}$.

Shew also that the following results are true:

3. $\dfrac{\sin^2\theta + \tan^2\theta - \tan^2\theta\cos^2\theta}{\cos^2\theta\,(1 + \cot^2\theta)} = \tan^4\theta.$

4. $\dfrac{(\cos A + \sin A)^3 + (\cos^2 A - \sin^2 A)\,(\cos A - \sin A)}{\cos A \sin A}$
$$= 2\,(\operatorname{cosec} A + \sec A).$$

5. $\cos^4 A - 1 + \sin^4 A - 2\cos^2 A \sin^2 A = 0.$

6. $(\sin a \cos\beta - \cos a \sin\beta)(\sin a \cos\beta + \cos a \sin\beta)$
$$= \sin^2 a - \sin^2\beta.$$

7. $(\cos a \cos\beta + \sin a \sin\beta)(\cos a \cos\beta - \sin a \sin\beta)$
$$= \cos^2\beta - \sin^2 a = \cos^2 a - \sin^2\beta.$$

8. $(\cos A + \sin A)(\cos A - \sin A) = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}.$

The denominator on the left being unity use Art. 24.

9. $\sin\phi\,(1 + \tan\phi) + \cos\phi\,(1 + \cot\phi) = \sec\phi + \operatorname{cosec}\phi.$

10. $\sin^2\beta + \operatorname{versin}^2\beta = 2\operatorname{versin}\beta.$

11. $\sec^2 A \operatorname{cosec}^2 A = \sec^2 A + \operatorname{cosec}^2 A.$

12. $\dfrac{\cos^2 A - \sin^2 B}{\sin^2 A \sin^2 B} + 1 = \cot^2 A \cot^2 B.$

13. $\cot^2 A \cos^2 A = \cot^2 A - \cos^2 A.$

14. $\dfrac{\tan A + \tan B}{\cot A + \cot B} = \tan A \tan B.$

Arts. 26, 27.

Find the other trigonometrical ratios, having given

(1) $\sin A = \dfrac{1}{2}.$ (2) $\cos A = \dfrac{5}{6}.$ (3) $\tan\theta = \dfrac{3}{4}.$

(4) $\cot\phi = \dfrac{10}{9}.$ (5) $\sec a = \sqrt{(2)}.$ (6) $\operatorname{cosec} a = 2\sqrt{(3)}.$

(7) $\operatorname{versin}\beta = \dfrac{3}{7}.$ (8) $\tan\phi = \dfrac{n}{m}.$ (9) $\sin^2\theta - \cos^2\theta = \dfrac{1}{2}.$

V. Arts. 28—30.

Find the value of the angle when

1. $\tan^2\theta - 1 = 2\tan\frac{\pi}{4}$.

By Art. 28 $\tan\frac{\pi}{4} = 1$.

Hence $\tan^2\theta - 1 = 2$, $\tan^2\theta = 3$, $\tan\theta = \sqrt{(3)}$,

and $\theta = 60°$ or $\frac{\pi}{3}$, by Art. 30.

2. $\tan\theta + \cot\theta = 4$.

$$\tan\theta + \frac{1}{\tan\theta} = 4.$$

$$\tan^2\theta + 1 = 4\tan\theta.$$

$\tan^2\theta - 4\tan\theta = -1$, a quadratic in $\tan\theta$.

$$\tan^2\theta - 4\tan\theta + 4 = -1 + 4 = 3.$$

$$\tan\theta = 2 \pm \sqrt{(3)}.$$

Equations of this sort very often reduce to quadratics.

Sometimes, however, we have one of this kind,

3. $(1 - 2\cos^2 a)(3 - \tan^2 a) = 0$.

This gives either

(i) $1 - 2\cos^2 a = 0$ or $\cos a = \frac{1}{\sqrt{(2)}}$, $a = 45°$ or $\frac{\pi}{4}$,

or (ii) $3 - \tan^2 a = 0$, $\tan a = \sqrt{(3)}$, $a = 60°$ or $\frac{\pi}{3}$,

for any value of a that satisfies (i) or (ii) must evidently satisfy the given equation.

4. Shew that the tangents of 30°, 45°, 60° are in Geometrical Progression.

5. Solve the following equations:

(1) $\tan^2 a + 4 \sin^2 a = 6$. (2) $\tan^2\phi + \cot^2\phi = 3\frac{1}{2}$.

(3) $\tan^2\phi + \cot^2\phi = 2$. (4) $\sin^2\theta + \cos\theta = \frac{5}{4}$.

(5) $\tan\theta = 2 \sin\theta$. (6) $\cos^2\theta - 2 \sin\theta + \frac{1}{4} = 0$.

(7) $3 \operatorname{cosec}^4\theta + 8 = 10 \operatorname{cosec}^2\theta$.

6. If $\cos 2A = \sin 3A$, find the number of degrees in A, see Arts. 18, 29.

7. Also when $\tan 8A = \cot A$.

8. Having given $\sin 36° = .58778$, shew that

$$\cos 36° = .80901.$$

9. Having given $\sin 50° = .76604$, shew that

$$\tan 50° = 1.19175.$$

10. Having given $\cot 1° = 57.28996$, shew that

$$\sin 1° = .01745.$$

11. Find $\sin 18\frac{1}{2}°$, having given

$\sin 18° = .30901$, $\sin 19° = .32557$, see end of Art. 30.

12. Find $\cos 29° 36' 12''$, having given

$$\sin 61° = .87462, \quad \sin 60° = \frac{\sqrt{(3)}}{2}.$$

VI., VII.

1. Find all the angles up to 540°, satisfying the relation $\sin\theta = \frac{1}{2}$, our first value is $\theta = 30°$.

But, by Arts. 35, 37, 38,

$$\begin{aligned}\sin\theta &= \sin(180° - \theta)\\ &= \sin(360° + \theta)\\ &= \sin(360° + 180° - A) \text{ or } \sin(540° - A).\end{aligned}$$

Hence our angles are 30°, 150°, 390°, 510°.

2. Find all the angles up to 540°, satisfying $\tan^2 A = 1$.

Here $\tan\theta = \pm 1$. Taking the positive sign.

(i) $\tan\theta = +1$, $\theta = 45°$, and the tangent is positive in the first and third quadrants, and the values recur after every complete revolution; our angles are therefore

$$45°,\ 180° + 45°,\ 360° + 45°.$$

(ii) $\tan\theta = -1$, the tangent is negative in the second and fourth quadrant. Hence

$180° - 45°$, $360° - 45°$, $360° + 180° - 45°$ are our angles.

Collecting all, they are 45°, 135°, 225°, 315°, 405°, 495°.

3. Find all the angles up to 4π, satisfying

$$\cos\theta = 0,\ \sin\theta = \frac{\sqrt{(3)}}{2},\ \sec^2\theta = 2,\ \tan\theta = \infty.$$

4. Determine the trigonometrical ratios for 585°.

This = 360° + 225°, and therefore we have only to find them for 225°, Art. 42.

Also for 690°, 930°.

5. Find the first six values for $\left\{n\pi + (-1)^n \frac{\pi}{6}\right\}$, where n is zero or a positive integer.

6. Trace the change in sign and magnitude from $\theta = 0$ to $\theta = 2\pi$ of

$$\cos\theta + \sin\theta,\ \tan\theta + \cot\theta.$$

7. (Art. 53). Solve the equations

$$\sin^2\theta + 2\cos^2\theta = 3,\ \sec^2\theta + \operatorname{cosec}^2\theta = \frac{1}{4}.$$

VIII.

1. Find the general value of θ when

$$2\cos^2\theta + 5\sin\theta - 4 = 0,$$

we have $$2 - 2\sin^2\theta + 5\sin\theta - 4 = 0,$$

$$\sin^2\theta - \frac{5}{2}\sin\theta = -1,$$

$$\sin^2\theta - \frac{5}{2}\sin\theta + \frac{25}{16} = -1 + \frac{25}{16} = \frac{9}{16},$$

$$\sin\theta = \frac{\pm 3 + 5}{4}.$$

By Art. 53. The only admissible value for θ is given by $\sin\theta = \frac{-3+5}{4} = \frac{1}{2}$. The least positive angle is therefore $\theta = 30° = \frac{\pi}{6}$, and our general formula is (Art. 56)

$$\theta = n\pi + (-1)^n \frac{\pi}{6}.$$

2. Sometimes in solving we come across such a solution as

$$\sin^2\theta = \frac{1}{2} \text{ or } \sin\theta = \pm\frac{1}{\sqrt{(2)}},$$

and taking both *signs* separately we should get two results

$$\theta = n\pi + (-1)^n \frac{\pi}{4}, \quad \theta = n\pi + (-1)^n \left(\pi + \frac{\pi}{4}\right).$$

Since the least positive angle for $\sin\theta = -\frac{1}{\sqrt{(2)}}$ is $\pi + \frac{\pi}{4}$.

We may, however, in such a case combine the two.

RULE. *In such cases write down all the angles up to* $2\pi + \theta$ (*i.e. one revolution after the first angle*) *that suit, observe what signs come before* θ, *and whether the multiples of* π *are even or odd, then make a suitable formula.*

Thus in our equation

$$\pi, \quad \pi - \frac{\pi}{4}, \quad \pi + \frac{\pi}{4}, \quad 2\pi - \frac{\pi}{4}, \quad 2\pi + \frac{\pi}{4},$$

all satisfy it. Now looking through these we see that odd multiples of π are followed both by $+\frac{\pi}{4}$ and $-\frac{\pi}{4}$, so also are even multiples. Whatever, therefore, the multiple be, there is always + *and* − following. Hence $\theta = n\pi \pm \frac{\pi}{4}$ is our general expression.

3. Find the general value of the angle from the following equations:

(1) $\sin\theta + \operatorname{cosec}\theta = 2$.

(2) $2\tan\theta = \dfrac{1}{\sin\theta\cos\theta}$, see Rule above.

(3) $3\sec^4\theta + 8 = 10\sec^2\theta$. (4) $\cos^2\theta = \cos^2 a$.

(5) $\tan^2\theta = \tan^2 a$. (6) $\sin^3\theta + \cos^3\theta = 0$.

(7) $5\sec^4\theta - 8\sec^2\theta - 48 = 0$.

4. What is the general value of θ when

$$(1)\ \operatorname{cosec}\theta = \frac{2}{\sqrt{(3)}}. \qquad (2)\ \tan^2\theta = \frac{1}{3}.$$

IX.

1. Prove the formula $\cot A + \tan A = 2\operatorname{cosec}2A$.

It is best as a general rule to take the more complicated side and transform by the aid of known formulæ into the other.

$$\cot A + \tan A = \frac{\cos A}{\sin A} + \frac{\sin A}{\cos A} = \frac{\sin^2 A + \cos^2 A}{\sin A\cos A}$$

$$= \frac{1}{\sin A\cos A}$$

$$= \frac{2}{\sin 2A}, \text{ Art. 77 (10)},$$

$$= 2\operatorname{cosec}2A.$$

2. Shew that $\tan A + \sec A = \tan\left(45^\circ + \frac{A}{2}\right)$.

Now $\tan\left(45^\circ + \frac{A}{2}\right) = \dfrac{1 + \tan\frac{A}{2}}{1 - \tan\frac{A}{2}}$, Art. 76,

$$= \frac{\sin\frac{A}{2} + \cos\frac{A}{2}}{\cos\frac{A}{2} - \sin\frac{A}{2}} = \frac{\left(\sin\frac{A}{2} + \cos\frac{A}{2}\right)^2}{\cos^2\frac{A}{2} - \sin^2\frac{A}{2}},$$

multiplying numerator and denominator by $\cos\frac{A}{2} + \sin\frac{A}{2}$,

$$= \frac{\sin^2\frac{A}{2} + \cos^2\frac{A}{2} + 2\sin\frac{A}{2}\cos\frac{A}{2}}{\cos^2\frac{A}{2} - \sin^2\frac{A}{2}}$$

$$= \frac{1 + \sin A}{\cos A}, \text{ Art. 77 (10) (11)},$$

$$= \sec A + \tan A.$$

3. Prove the formulæ:

(1) $\operatorname{cosec} 2A + \cot 2A = \cot A$.

(2) $\sec A = 1 + \tan A \tan\frac{A}{2}$.

(3) $\operatorname{cosec} 2A = \dfrac{1 + \cot^2 A}{2\cot A}$. (4) $\cos A = \cos^4\frac{A}{2} - \sin^4\frac{A}{2}$.

(5) $2\operatorname{cosec} 2A = \sec A \operatorname{cosec} A$.

(6) $\dfrac{2\sin A + \sin 2A}{2\sin A - \sin 2A} = \cot^2\frac{A}{2}$.

(7) $2\sin^2 A \sin^2 B + 2\cos^2 A \cos^2 B = 1 + \cos 2A \cos 2B$.

(8) $\cos^2(A-B)+\cos^2 B-2\cos(A-B)\cos A\cos B=\sin^2 A.$

(9) $\sin^2(A-B)+\sin^2 B+2\sin(A-B)\sin B\cos A=\sin^2 A.$

(10) $$\frac{1-\tan^2\left(\frac{\pi}{4}-\theta\right)}{1+\tan^2\left(\frac{\pi}{4}-\theta\right)}=\sin 2\theta.$$

(11) $$\frac{4\tan\theta-4\tan^3\theta}{(1+\tan^2\theta)^2}=\sin 4\theta.$$

4. Shew $\dfrac{\cos 2A-\cos 4A}{\cos 2A+\cos 4A}=\tan 3A\tan A,$

$$\frac{\cos 2A-\cos 4A}{\cos 2A+\cos 4A}=\frac{2\sin\frac{2A+4A}{2}\sin\frac{4A-2A}{2}}{2\cos\frac{2A+4A}{2}\cos\frac{4A-2A}{2}},\text{ Art. 72 (7)(8)},$$

$$=\frac{\sin 3A\sin A}{\cos 3A\cos A}=\tan 3A\tan A.$$

5. Prove the formulæ:

(1) $\dfrac{\sin A+\sin 3A}{\cos A+\cos 3A}=\tan 2A.$

(2) $\dfrac{\sin A+2\sin 5A+\sin 9A}{\cos A+2\cos 5A+\cos 9A}=\tan 5A.$

(3) $\cos A+\cos(A+2B)-2\cos(A+B)\cos B=0.$

(4) $\sin 5A\sin A=\sin^2 3A-\sin^2 2A.$

(5) $\sin 7A\sin 3A=\sin^2 5A-\sin^2 2A.$

(6) $\sin 3A\sin^3 A+\cos 3A\cos^3 A=\cos^3 2A.$

(7) $\tan 2A-\tan A=\dfrac{2\sin A}{\cos A+\cos 3A}.$

(8) $\sin 3A\operatorname{cosec} A-\cos 3A\sec A=2.$

(9) $\dfrac{\sin B}{\sin A}=\dfrac{\sin(2A+B)}{\sin A}-2\cos(A+B).$

(10) $\cos A + \cos(120° - A) + \cos(120° + A) = 0.$

(11) $\dfrac{\cos 2A \pm \cos 2nA + \cos 2(2n-1)A}{\sin 2A \pm \sin 2nA + \sin 2(2n-1)A} = \cot 2nA.$

(12) $\cos^2 2A - \cos^2 3A = \sin A \sin 5A.$

6. Shew $4 \sin A \sin(60° - A) \sin(60° + A) = \sin 3A.$

$$4 \sin A \sin(60° - A) \sin(60° + A)$$

$$= 4 \sin A \,.\, \frac{1}{2}\{\cos 2A - \cos 120°\}, \text{ Arts. 72, 73, 74,}$$

$$= 2 \sin A \left\{1 - 2\sin^2 A + \frac{1}{2}\right\}, \text{ Arts. 77 (12), 30,}$$

$$= 3 \sin A - 4 \sin^3 A$$

$$= \sin 3A, \text{ Art. 78.}$$

7. Prove the following:

(1) $4 \cos A \cos(120° - A) \cos(120° + A) = \cos 3A.$

(2) $\tan\left(15° - \dfrac{A}{2}\right) \cot\left(15° + \dfrac{A}{2}\right) = \dfrac{1 - \sin A}{1 + \sin A}.$

8. Solve the following equations:

(1) $\sin 8\theta + \sin\theta = 0.$ (2) $\cos 3\theta + \cos 2\theta + \cos\theta = 0.$

(3) $\tan\theta + \tan\left(\dfrac{\pi}{4} + \theta\right) = 2.$ (4) $\cot\left(\dfrac{\pi}{4} + \theta\right) = 3\cot\left(\dfrac{\pi}{4} - \theta\right).$

(5) $\sin\dfrac{7\theta}{2} - \sin\dfrac{\theta}{2} = \sin\dfrac{3\theta}{2}.$ (6) $4 \sin\theta \sin 3\theta = 1.$

9. If A, B, C be the angles of a triangle, then

$$\sin A + \sin B + \sin C = 4 \cos\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}.$$

$$\sin A + \sin B + \sin C = 2 \sin\frac{A+B}{2} \cos\frac{A-B}{2} + 2 \sin\frac{C}{2} \cos\frac{C}{2}.$$

Now $A + B + C = 180°$, and $\frac{C}{2} = 90° - \frac{A+B}{2}$,

and the above

$$= 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} + 2 \sin \frac{A+B}{2} \cos \frac{A+B}{2} \text{ (Art. 18)}$$

$$= 2 \sin \frac{A+B}{2} \left\{ \cos \frac{A-B}{2} + \cos \frac{A+B}{2} \right\}$$

$$= 2 \cos \frac{C}{2} . 2 \cos \frac{A}{2} \cos \frac{B}{2} = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}.$$

10. If $A + B + C = 180°$, shew

(1) $\cos A + \cos B + \cos C = 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} + 1.$

(2) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.$

(3) $\sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} + 2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = 1.$

(4) $\cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C - 1 = 0.$

(5) $1 + \cos A \cos B \cos C$

$= \cos A \sin B \sin C + \cos B \sin A \sin C + \cos C \sin A \sin B.$

(6) $\tan A + \tan B + \tan C = \tan A \tan B \tan C.$

(7) $\cot A \cot B + \cot A \cot C + \cot B \cot C = 1.$

(8) $\cos \frac{A}{2} + \cos \frac{B}{2} + \cos \frac{C}{2} = 4 \cos \frac{\pi - A}{4} \cos \frac{\pi - B}{4} \cos \frac{\pi - C}{4}.$

(9) $\sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2} - 1$

$$= 4 \sin \frac{\pi - A}{4} \sin \frac{\pi - B}{4} \sin \frac{\pi - C}{4}.$$

11. If $A + B + C = 90°$.

(1) $\tan A \tan B + \tan A \tan C + \tan B \tan C = 1.$

(2) $\cot A + \cot B + \cot C = \cot A \cot B \cot C.$

(3) $\sin 2A + \sin 2B + \sin 2C = 4 \cos A \cos B \cos C.$

(4) $\dfrac{\cos A + \sin C - \sin B}{\cos B + \sin C - \sin A} = \dfrac{1 + \tan\dfrac{A}{2}}{1 + \tan\dfrac{B}{2}}$.

12. If in any triangle ABC, $\sin A + \sin C = 2 \sin B$.

Shew that $\cot\dfrac{A}{2}$, $\cot\dfrac{B}{2}$, $\cot\dfrac{C}{2}$ are in Arithmetical Progression.

13. $A + B + C = \pi$.

Shew

$$\frac{\tan A + \tan B + \tan C}{(\sin A + \sin B + \sin C)^2} = \frac{\tan\dfrac{A}{2}\tan\dfrac{B}{2}\tan\dfrac{C}{2}}{2\cos A\cos B\cos C} \text{ (see Ex. 9).}$$

14. Given $\sin 45° = \dfrac{1}{\sqrt{2}}$, $\cos 30° = \dfrac{1}{2}$.

Find

$\sin 15°$, $\cos 15°$, $\sin 75°$, $\cos 75°$, $\tan 7\frac{1}{2}°$, $\sin 22\frac{1}{2}°$, $\cos 11\frac{1}{4}°$.

X.

1. Find the characteristic of .6, .593, .80764, .003 to the base 10, Art. 95.

Also of 16, 375, 8974, .64 to the base 7.

2. Given $\log_{10} 36702 = 4.5646897$.

Write down the logarithms of

3670200, 3670.2, 3.6702, .036702, .0036702, Art. 96.

3. Given $\log 2 = .30103$, $\log 7 = .845098$; find $\log 98$.

Now $\quad 98 = 7 \times 14 = 7 \times 7 \times 2 = 7^2 \times 2$.

Hence $\log 98 = \log(7^2 \times 2) = \log 7^2 + \log 2$ (Art. 87)

$= 2 \log 7 + \log 2$ (Art. 91)

$= 1.690196 + .301030$

$= 1.991226$.

4. Given $\log 2 = .30103$. Find $\log \sqrt[3]{(2)}$.

$$\log \sqrt[3]{(2)} = \log (2)^{\frac{1}{3}} = \frac{1}{3} \log 2 \text{ (Art. 91)}$$
$$= .100343.$$

5. Given $\log 2 = .301030$, $\log 3 = .477121$, $\log 7 = .845098$. Find logarithms of

$$6,\ 15,\ 5.4,\ 17.5,\ .875,\ 6860.$$

6. Find logarithm of 81 to base $\sqrt{(3)}$.

Let $\log_{\sqrt{(3)}} 81 = x$. Then $\sqrt{(3)}^x = 81$ (by Art. 83),

or $\qquad 3^{\frac{x}{2}} = 81 = 3^4$, whence $\frac{x}{2} = 4$, or $x = 8$,

and $\qquad \log_{\sqrt{(3)}} 81 = 8.$

7. Find logarithm of 256 to base $2\sqrt{(2)}$, of 128 to base $\sqrt[3]{(4)}$, of 243 $\sqrt[3]{(9)}$ to base $\sqrt{(3)}$.

8. Given $\log 2 = .30103$; find logarithms of

$$5,\ 125,\ .125,\ \sqrt[15]{\left(\frac{5^{90}}{2^{60}}\right)},\ \sqrt[14]{\left(\frac{2^{60}}{5^{30}}\right)}.$$

Also find $\qquad \log \cos 45°,\ \log \tan \frac{5\pi}{4}.$

9. Given $\log 18 = 1.255272$, $\log 25 = 1.397940$. Find those of 2, 3, 16, 450, .075, 3.75.

10. Find x when $\left(\frac{1}{2}\right)^x = \frac{2}{3}$.

Given $\qquad \log 2 = .301030$, $\log 3 = .477121$.

Taking logarithms of both sides (Art. 98)

$$\log \left(\frac{1}{2}\right)^n = \log \frac{2}{3},$$

or $$x(\log 1 - \log 2) = \log 2 - \log 3 \text{ (Art. 88)}$$

$$-x \log 2 = \log 2 - \log 3 \text{ (Art. 85)},$$

$$(x+1)\log 2 = \log 3,$$

$$x + 1 = \frac{477121}{301030},$$

$$x = \frac{176091}{301030} = .5849.$$

11. Find x when $20^x = 100$.

12. Given $\log 3$ and $\log 2$ (see above). Find x when

$$(4\tfrac{1}{2})^{(3.5)x} = 1000.$$

13. Find x when $5^{5-4x} = 2^{x+3}$, $\log 2$ being supposed known.

XI. Arts. 101—105.

1. If ABC be a right-angled triangle, $C = 90°$, shew that

$$\cos 2A = \frac{b^2 - a^2}{b^2 + a^2}.$$

Construct the figure.

Then $\cos 2A = \cos^2 A - \sin^2 A$

$$= \frac{b^2}{c^2} - \frac{a^2}{c^2} = \frac{b^2 - a^2}{c^2} = \frac{b^2 - a^2}{b^2 + a^2} \text{ (Euc. I. 47)}.$$

2. With the same conditions as in (1). Shew that

(1) $\cos(A - B) = \dfrac{2ab}{c^2}$. (2) $\cos(45° \pm A) = \dfrac{1}{\sqrt{(2)}}\left(\dfrac{b \mp a}{c}\right)$.

(3) $\tan 2A = \dfrac{2ab}{b^2 - a^2}$. (4) $\tan \dfrac{1}{2} A = \left(\dfrac{c - b}{c + b}\right)^{\frac{1}{2}}$.

3. In any triangle ABC, shew

$$\tan \frac{1}{2} A \tan \frac{1}{2} B = \frac{a + b - c}{a + b + c}.$$

By Art. 104,

$$\tan\frac{1}{2}A\tan\frac{1}{2}B = \sqrt{\left\{\frac{(s-b)(s-c)}{s.s-a}\right\}}\cdot\sqrt{\left\{\frac{(s-a)(s-c)}{s.s-b}\right\}}$$

$$= \frac{s-c}{s} = \frac{\dfrac{a+b-c}{2}}{\dfrac{a+b+c}{2}}$$

$$= \frac{a+b-c}{a+b+c}.$$

4. In any triangle ABC, prove

(1) $a = b\cos C + c\cos B$. (2) $a(b\cos C - c\cos B) = b^2 - c^2$.

(3) $\tan B = \dfrac{b\sin C}{a - b\cos C}$. (4) $\dfrac{\tan B}{\tan C} = \dfrac{a^2+b^2-c^2}{a^2+c^2-b^2}$.

(5) $\dfrac{\sin(A-B)}{\sin(A+B)} = \dfrac{a^2-b^2}{c^2}$. (6) $\dfrac{\text{versin}\,A}{\text{versin}\,B} = \dfrac{a(a+c-b)}{b(b+c-a)}$.

(7) $1 - \tan\dfrac{A}{2}\tan\dfrac{B}{2} = \dfrac{2c}{a+b+c}$.

5. If the sides a, b of a triangle include an angle of 120°. Shew $c^2 = a^2 + ab + b^2$ (Art. 103).

6. If $2\cos B = \dfrac{\sin A}{\sin C}$, the triangle is isosceles.

7. If the sides of a triangle are in Arithmetical Progression, so also are the opposite sides.

8. If $\dfrac{\tan A}{\tan B} = \dfrac{\sin^2 A}{\sin^2 B}$, the triangle is isosceles or right angled.

9. If in a triangle $A : B : C :: 2 : 3 : 4$, then will

$$\cos\frac{1}{2}A = \frac{a+c}{2b}.$$

10. In any triangle, if p be the perpendicular from A upon BC,

$$p = \frac{b^2 \sin C + c^2 \sin B}{b + c}.$$

XII. Arts. 107—110.

1. If $A + B = C$ in a triangle, and $a = \sqrt{(3b)}$. Solve the triangle.

Since $$A + B + C = 180°,$$

we have $$2C = 180°, \quad C = 90°, \quad A + B = 90°.$$

Now $$\frac{\sin A}{\sin B} = \frac{a}{b} = \sqrt{(3)},$$

and $$\sin B = \sin(90° - A) = \cos A.$$

Hence $$\tan A = \sqrt{(3)}, \quad A = 60°, \quad B = 30°.$$

2. Solve the triangle, having given

(1) $c = 75$, $A = 30°$, $C = 90°$.

(2) $c = 50$, $a = 25$, $C = 90°$.

(3) $a = 80$, $B = 15°$, $C = 90°$, Ex. IX. 14.

3. Solve the triangle when

$$C = 120, \quad B = 36°, \quad C = 90°,$$

having given $\sin 36° = .5877853$, $\sin 54° = .8090170$.

4. Three towns A, B, C are equidistant from each other. D is mid-way between A and B. Compare the distances of D from A and C.

5. Determine the altitude of the sun when the length of the shadow of a vertical stick is to the length of the stick as $1 : \sqrt{(3)}$.

6. What is the height of a tower whose top appears at an elevation of 30° to an observer 140 feet from the foot of the tower, his eye being 5 feet from the ground?

7. A ladder is just long enough when vertical to reach to the top of a building 60 feet high. If it be placed against the building so as to reach to a point 30 feet from the top, find the angle of elevation.

8. A ladder 38 feet long reaches to a window 19 feet from the ground. It is turned over and just reaches a window on the opposite side of the street 19 $\sqrt{(2)}$ feet high. The foot of the ladder is supposed to remain stationary. Find the width of the street.

9. A man observes that the elevation θ of a hill is such that $\tan\theta = \dfrac{\sqrt{(2)} - 1}{\sqrt{(3)} + \sqrt{(2)}}$. If he be one mile from the foot, what is the height?

10. A, B, C are three vessels. The distance between A and C is always 10 miles, and the lines joining A and B, C and B at any moment form a right angle. If the distance between A and B is five miles, determine C's position.

XIII. Arts. 111—115.

1. If $a = 80$, $b = 240$, $A = 30°$, is there any ambiguity?

Here $\dfrac{b}{a}\sin A = \sin B = \dfrac{240}{80}\sin 30° = 3 \times \dfrac{1}{2} = 1\frac{1}{2}$,

which is impossible. The triangle does not exist, Art. 113 (a).

2. Find A having given $\sin B = .5$, $a = 10$, $b = 5$.

3. Having given

$$\sin 18° = \frac{\sqrt{(5)} - 1}{4} = \sin A, \quad a = 4, \quad b = 4 + \sqrt{(80)},$$

solve the triangle.

4. If B_1, C_1, B_2, C_2, be the angles of the two triangles in the ambiguous case,

$$\frac{\sin C_1}{\sin B_1} + \frac{\sin C_2}{\sin B_2} = 2 \cos A.$$

5. Solve the triangle when

$$\log a + \log 10 = \log b + L \sin A, \text{ Art. 99.}$$

6. The sides of a triangle are 32, 40, 66; find the greatest angle, having given $\log 207 = 2.3159703$.

$L \cot 66° \, 17' \, 16'' = 9.6426853$, $\log 1073 = 3.0305997$,

$$\cot \frac{C}{2} = \sqrt{\left\{\frac{s(s-c)}{(s-a)(s-b)}\right\}}, \text{ Art. 104,}$$

$$2s = 32 + 40 + 66 = 138, \quad s = 69,$$

$$\cot \frac{C}{2} = \sqrt{\left(\frac{69.3}{37.29}\right)} = \left(\frac{207}{1073}\right)^{\frac{1}{2}},$$

$$\begin{aligned} L \cot \frac{C}{2} &= 10 + \frac{1}{2} \log 207 - \frac{1}{2} \log 1073 \\ &= 10 + 1.1579851 - (1.5152999) \\ &= 9.6426853 \\ &= L \cot 66° \, 17' \, 16'' \text{ by what is given,} \end{aligned}$$

$$\frac{C}{2} = 66° \, 17' \, 16'', \quad C = 132° \, 34' \, 32''.$$

7. In a triangle ABC, when $a = 18$, $b = 20$, $c = 22$, find $L \tan \frac{A}{2}$, having given

$$\log 3 = .4771213 \text{ and } \log 5 = .69897.$$

8. Apply the formula for $\cos \frac{A}{2}$ to find the greatest angle in a triangle, whose sides are 5, 6, 7 feet, having given $\log 6 = .7781513$, $L \cos 39° \, 13' \, 53.5'' = 9.8890756$ (Euc. I. 19).

9. Given $a = 18$, $b = 2$, $C = 55°$,

$L \tan 113° 53' 41'' = 10.1866133$, $L \cot 27° 30'' = 10.2835233$,

(Art. 112). Solve the triangle.

10. If $a = 2$, $c = 3$, $L \sin A = 9.5228787$.

Find C, when $\log 3 = .4771213$.

XIV.

1. ABC is a triangle, its sides are 3, 4, 5 feet respectively, find the area.

$$\text{area} = \sqrt{\{s(s-a)(s-b)(s-c)\}}, \text{ Art. 116 (2)},$$

$$2s = 12, \quad s = 6,$$

$$A = \sqrt{(6.3.2.1)} = 6 \text{ square feet.}$$

Or, since $3^2 + 4^2 = 5^2$, the triangle is right angled, and, (Art. 116 (1)),

$$\text{area} = \frac{1}{2} ab \sin C = \frac{1}{2}.3.4 \sin 90° = \frac{12}{2} = 6.$$

2. Two sides of a triangle are as 1 : 4, and the included angle = 30°. Find the area.

3. Find the area when $a = 16$, $b = 20$, $c = 24$.

4. Shew that the sum of the two areas in the *ambiguous case* (Art. 113) $= b^2 \sin A \cos A$.

5. Find the area of the inscribed circle in Ex. (3).

6. Find the area of each of the escribed circles of Ex. (3).

7. Three tobacco pipes, each 14 inches long, are made into a triangular stand. Allowing 2 inches for the overlapping in each pipe, find the radius of the bottom of the smallest circular vessel that can stand on them.

8. A triangular piece of ground, the sides of which are 300, 600, 700 yards long respectively, is enclosed by a circular fence. Find the length of the fence. (Circumference of circle = $2\pi \times$ radius).

9. Find the area of a regular hexagon described about a circle, each side being 6 feet (Art. 123).

10. The sides of a triangle are 3, 5, 6; find the radii of the inscribed and circumscribed circles.

11. Find the ratio between (1) the sides, (2) the areas, of an equilateral triangle, and a square inscribed in the same circle (radius r).

12. If in a regular polygon of n sides, each side is $2a$; the sum of the radii of the circumscribed and inscribed circles is $a.\cot\frac{\pi}{2n}$.

XV. Arts. 125, 126.

1. Find the area of the circle described in an equilateral triangle whose sides each $= a$, and shew that its circumference $= \frac{\pi a}{\sqrt{(3)}}$.

2. A man has a square plot of ground 3600 square yards in extent. He turfs the largest circle he can within it. Find the area of what is left for planting.

3. The radius of the earth is about 8000 miles, find the area of the parallels of latitude 30°, 45°, 60°.

4. With the angular points of an equilateral triangle as centres, and a radius equal to half the side, arcs of circles are described touching each other: determine the area of the figure they form (side of triangle $= a$).

ANSWERS.

I.

2. $125^g\,56'\,63''.58$, $87^g\,60'\,53''.39$, $62^g\,53'\,15''$, $0^g\,60'\,41''.6$.

4. $113°\,0'\,35''$, $78°\,50'\,41''.3$, $56°\,16'\,42''$, $0°\,36'\,15''$.

5. $63°$, $9°$. 6. $4\frac{1}{3}$. 8. $90°$. 9. $55°$, $35°$.

II.

2. (1) .99484. (2) 1.0035. (3) $\frac{\pi}{10}$. (4) $\frac{\pi}{8}$. (5) $\pi+\frac{\pi}{4}$.

4. $57°\,17'\,44''.8$, $28°\,38'\,52''.4$, $85°\,56'\,37''.2$,
$19°\,5'\,54''.9$, $38°\,11'\,49''.8$, $47°\,44'\,47''.2$.

5. $57^g\,29'\,57''$, $28^g\,64'\,79''$, $85^g\,94'\,36''$, $19^g\,9'\,86''$.
$38^g\,19'\,71''$, $47^g\,74'\,64''$.

6. $9°\,54'\,93''$. 7. $81°$, $60°$, $39°$. 8. $\frac{180°}{7}$, $\frac{360°}{7}$, $\frac{720°}{7}$.

III.

1. $50°\,44'\,32''$, $15°\,43'\,57''$, $163°\,56'\,33''$, $3°\,40'\,39''$.

2. $\tan A \cot B$.

3. $(\sin\theta + \cos\theta + \tan\theta)(\sin\theta - \cos\theta - \tan\theta)$
$= \sin^2\theta - (\cos\theta + \tan\theta)^2$.

4. $\sin\beta = \pm\frac{1}{\sqrt{(3)}}$.

IV.

2. (1) $\sin A = \pm\frac{\sqrt{(3)}}{2}$. (2) $\cos A = \frac{\sqrt{(31)}}{6}$.

(3) $\cos\theta = \frac{4}{5}$. (4) $\sin\phi = \frac{9}{\sqrt{(181)}}$.

(5) $\cos a = \frac{1}{\sqrt{(2)}}$. (6) $\sin a = \frac{1}{2\sqrt{(3)}}$.

(7) $\cos\beta = \frac{4}{7}$. (8) $\sin\gamma = \frac{n}{\sqrt{(n^2 + m^2)}}$.

(9) $\sin\theta = \frac{\sqrt{(3)}}{2}$.

V.

5. (1) $a = \frac{\pi}{3}$. (2) $\phi = \frac{\pi}{3}$ or $\frac{\pi}{6}$. (3) $\phi = \frac{\pi}{4}$. (4) $\theta = \frac{\pi}{3}$.

(5) $\theta = 0$ or $\frac{\pi}{3}$. (6) $\theta = \frac{\pi}{6}$. (7) $\theta = \frac{\pi}{4}$ or $\frac{\pi}{3}$.

6. 18°. 7. 10°.

VI., VII.

3. $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$, *i.e.* $(2n+1)\frac{\pi}{2}$, where n is zero or any integer.

$\frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}$. All multiples of $\frac{\pi}{4}$ up to and including $\frac{15\pi}{4}$. Same as for $\cos\theta = 0$.

4. Same as for 330°, 210°.

5. 30°, 150°, 210°, 330°, 420°, 510°. 7. Both impossible.

VIII.

3. (1) $\theta = n\pi + (-1)^n \frac{\pi}{2}$. (2) $\theta = (2n+1)\frac{\pi}{2}$ and $n\pi \pm \frac{\pi}{4}$.

(3) $\theta = n\pi \pm \frac{\pi}{4}$, $\theta = n\pi \pm \frac{\pi}{6}$. (4) $\theta = n\pi \pm a$.

(5) $\theta = n\pi \pm a$. (6) $\theta = n\pi - \frac{\pi}{4}$. (7) $n\pi \pm \frac{\pi}{3}$.

4. (1) $\theta = n\pi + (-1)^n \frac{\pi}{3}$. (2) $\theta = n\pi \pm \frac{\pi}{6}$.

IX.

8. (1) $\frac{9\theta}{2} = n\pi$, $\frac{7\theta}{2} = \frac{n\pi}{2}$.

(2) $2\theta = \left(n + \frac{1}{2}\right)\pi$ or $\theta = 2n\pi \pm \frac{2\pi}{3}$.

(3) $2\theta = n\pi + (-1)^n \frac{\pi}{6}$. (4) $\frac{\pi}{4} + \theta = n\pi \pm \frac{\pi}{6}$.

(5) $\frac{3\theta}{2} = n\pi$, $2\theta = 2n\pi \pm \frac{\pi}{3}$.

(6) $\theta = \frac{\pi}{10}$ or $\frac{3\pi}{10}$, see Art. 81.

14. $\sin 15^\circ = \frac{\sqrt{(3)} - 1}{2\sqrt{(2)}} = \cos 75^\circ$, $\cos 15^\circ = \frac{\sqrt{(3)} + 1}{2\sqrt{(2)}} = \sin 75^\circ$,

$\tan 7\frac{1}{2}^\circ = \frac{\sqrt{(2)} - 1}{\sqrt{(2)} + \sqrt{(3)}}$, $\sin 22\frac{1}{2}^\circ = \frac{1}{2}\{2 - \sqrt{(2)}\}^{\frac{1}{2}}$,

$\cos 11\frac{1}{4}^\circ = \frac{1}{2}[2 + \{2 + \sqrt{(2)}\}^{\frac{1}{2}}]^{\frac{1}{2}}$.

X.

1. 0, 2, 4, − 3, 1, 3, 5, − 1.

2. 6.5646897, 3.5646897, .5646897, $\bar{2}$.5646897, $\bar{3}$.5646897.

5. $\log 6 = .778151$, $\log 15 = 1.176091$, $\log 5.4 = .732393$, $\log 17.5 = 1.243038$, $\log .875 = \bar{1}.942008$, $\log 6860 = 3.836324$.

7. $5.\dot{3}$, $\frac{21}{2}$, $243\sqrt[3]{(9)} = \{\sqrt{(3)}\}^{\frac{34}{3}}$.

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9. $\log 2 = .30103$, $\log 3 = .4771213$, $\log .16 = \bar{1}.204120$, $\log 450 = 2.653212$, $\log .075 = \bar{2}.875061$, $\log 3.75 = .574031$.

11. 1.537. 12. 1.312. 13. 1.06.

XII.

2. (1) $a = 37\frac{1}{2}$, $b = 75 \times \frac{\sqrt{(3)}}{2}$, $B = 60°$.

(2) $B = 60°$, $A = 30°$, $b = 25\sqrt{(3)}$.

(3) $c = 80\{\sqrt{(6)} - \sqrt{(2)}\}$, $b = 80\{2 - \sqrt{(3)}\}$.

3. $a = 97.082040$, $b = 70.534236$. 4. $1 : \sqrt{(3)}$.

5. 60°. 6. $\frac{140 + 5\sqrt{(3)}}{\sqrt{(3)}}$ feet. 7. 30°.

8. $19\{\sqrt{(3)} + \sqrt{(2)}\}$ feet.

9. .1312 of a mile = 692 feet nearly.

10. $5\sqrt{(3)}$ miles from B.

XIII.

2. 90°. 3. $B = 90°$, $C = 72°$, $c = 4\sqrt{[\{5 + 2\sqrt{(5)}\}]}$.

5. $A = 90°$. 7. 9.6733937. 8. 78° 27′ 47″.

9. 119° 26′ 51″, 5° 33′ 9″. 10. 30° or 150°.

XIV.

2. 1. 3. $60\sqrt{(7)}$. 5. $2\sqrt{(7)}$.

6. $\frac{30}{\sqrt{(7)}}$, $6\sqrt{(7)}$, $10\sqrt{(7)}$. 7. $2\sqrt{(3)}$ inches.

8. $R = \frac{315\sqrt{(5)}}{2}$ yards circumference $= \pi\{315\sqrt{(5)}\}$.

9. $6\sqrt{(3)}$ square yards.

10. $r = \frac{16}{56}\sqrt{(14)}$, $R = \frac{45}{56}\sqrt{(14)}$.

11. $\sqrt{(3)} : \sqrt{(2)}$ or $3^{\frac{3}{2}} : 8$.

XV.

2. 772.65 square yards.

3. 48000000, 32000000, 16000000 square miles.

4. $\frac{\pi a^2}{8}$.

THE END.

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[JANUARY 1868.]

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